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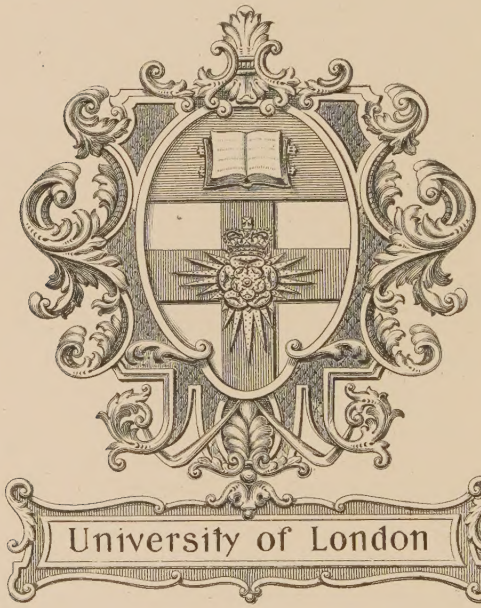

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. X.

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LONDON MATHEMATICAL SOCIETY.

VOL. X.

FIFTEENTH SESSION, 1878–79.

November 14th, 1878.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

Lord RAYLEIGH, F.R.S., President, in the Chair.

The Treasurer (S. Roberts, Esq., F.R.S.) read his Report, which, on the motion of Prof. Cayley, F.R.S., seconded by Prof. W. G. Adams, F.R.S., was accepted.

At the request of the Chairman, Prof. Adams consented to act as Auditor.

The Secretaries' Report was read and adopted.

From the Report of the Secretaries, it appeared that the number of Members during the past session had been 137, and the number of Foreign Members 12, and that the present number of Life Members is 43.

The communications made to the Society during the past Session had been as follows:

"On the function $\phi(x) = \dfrac{ax+b}{cx+d}$," and "On the Theta-Functions:" Prof. Cayley, F.R.S.

"The Calculus of Equivalent Statements (i., ii.):" Mr. H. McColl, B.A.

"Progressive Waves:" Lord Rayleigh, F.R.S.

"On the Triple Generation of Three-Bar Curves," "On the Mass-Centre of an Octahedron," and "On Vortex Motion:" Prof. Clifford, F.R.S.

"Notes on Normals:" Mr. S. Roberts, F.R.S.

"On the Geometrical Representation of Imaginary Quantities, and the Real (m, n) Correspondence of two Planes:" Prof. Cayley, F.R.S.

"On the meaning of the Differential Symbol D^n (where n is fractional):" Mr. J. Hammond, M.A.

"On Partial Differential Equations with several Dependent Variables:" Prof. Lloyd Tanner, M.A.

"On the Relations between the Functions of Laplace and Bessel:" Lord Rayleigh, F.R.S.

"On a general method of Solving Partial Differential Equations:" Prof. Tanner, M.A.

"On the Conditions of Steady Motion of a Fluid;" Prof. H. Lamb, M.A.

"On a Property of the Four-piece Linkage," and "On a curious Locus in Linkages:" Mr. A. B. Kempe, B.A.

"On Mr. Robert Flower's new mode of Computing Logarithms:" Mr. S. M. Drach, F.R.A.S.

"On the Plückerian Characteristics of the Modular Equations (i., ii.):" Prof. H. J. S. Smith, F.R.S.

"On the Electrical Capacity of a long narrow Cylinder and of a Disk of sensible thickness:" Prof. J. Clerk Maxwell, F.R.S.

"Note relating to the Theory of the Division of the Circle:" Prof. H. J. S. Smith, F.R.S.

"On the Astatic Conditions of a Body acted on by given Forces:" Prof. G. M. Minchin, M.A.

"On certain Extensions of Frullani's Theorem:" Mr. C. Leudesdorf, M.A.

"On the Transformation of Elliptic Functions:" Dr. Klein, of Munich.

"Theory of Groups:" Prof. Cayley, F.R.S.

"Notes on the Solution of Statical Problems connected with Linkworks and other Plane Mechanisms:" Prof. A. B. W. Kennedy.

"A Generalized Form of certain Series:" Mr. J. W. L. Glaisher, F.R.S.

"Conjugate Four-piece Linkages:" Mr. A. B. Kempe, B.A.

"The Characteristics of Systems of Conics:" M. Halphen.

"A method in the Analysis of Plane Curves:" Mr. J. J. Walker, M.A.

"The Flexure of Spaces:" Mr. C. J. Monro, B.A.

"The Decomposition of certain Numbers into Sums of two Square Integers by Continued Fractions:" Mr. S. Roberts, F.R.S.

"A new method of finding Differential Resolvents of Algebraical Equations:" Mr. R. Rawson.

The following Mathematical Journals had been subscribed for:

"Nouvelles Annales de Mathématique;" Paris.

"Giornale di Matematica;" Napoli.

"Mathematische Annalen;" Leipzig.

"Philosophical Magazine and Journal of Science;" London.

"Quarterly Journal of Mathematics;" Cambridge.

"Messenger of Mathematics;" Cambridge.

The following addition had been made to the number of Journals presented in the previous Session:

"The American Journal of Pure and Applied Mathematics;" Baltimore.

The Meeting then proceeded to the election of the new Council. The Scrutators (Messrs. C. Pendlebury and R. F. Scott) having examined the balloting lists, declared the following gentlemen (nominated by the Council) duly elected:

President, Mr. C. W. Merrifield, F.R.S.; Vice-Presidents, Prof. Cayley, F.R.S., Lord Rayleigh, F.R.S.; Treasurer, Mr. S. Roberts, F.R.S.; Hon. Secretaries, Messrs. Morgan Jenkins, M.A., and Robert Tucker, M.A.; other Members, Mr. J. W. L. Glaisher, F.R.S., Mr. Harry Hart, M.A., Prof. Henrici, F.R.S., Dr. Hirst, F.R.S., Dr. J. Hopkinson, F.R.S., Mr. A. B. Kempe, B.A., Dr. Spottiswoode, F.R.S., Prof. H. J. S. Smith, F.R.S., Mr. H. M. Taylor, M.A., and Mr. J. J. Walker, M.A.

Mr. C. W. Merrifield, F.R.S., the newly-elected President, then took the Chair.

Mr. J. D. H. Dickson, M.A., was elected a Member, and Prof. W. Stanley Jevons, F.R.S., was proposed for election. The Rev. A. Freeman, M.A., and Prof. A. W. Reinold, M.A., were admitted into the Society.

The Chairman read, and laid before the Meeting, a letter received from Mr. Warren de la Rue, F.R.S., respecting a memorial to M. Leverrier.

The following communications were made to the Society:

"On the Instability of Jets:" Lord Rayleigh, F.R.S.

"On Self-strained Frames of six Joints:" Prof. Crofton, F.R.S.

"On Equivalent Statements (iii.):" Mr. Hugh McColl, B.A. (Abstract read by Mr. Tucker.)

The following presents were made to the Society:

Carte de Visite likenesses of Dr. P. Gordan and M. Mannheim.

"Bulletin de la Société Mathématique de France," Tome vi., No. 6 et dernier; Paris, 1878.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^{me} Série, Tome ii., Juillet 1878; Paris, 1878.

"Nautical Almanac for 1882;" presented by Lords Commissioners of the Admiralty.

"Educational Times," Nov. 1878.

"Annuaire de l'Académie Royale," 1877–8.

Bulletins de l'Académie Royale: 44^{me} Année, 2^{me} Série, Tome xli., 1876; Tome xlii., 1876. 46^{me} Année, 2^{me} Série, Tome xliii., 1877; Tome xliv., 1877. 47^{me} Année, 2^{me} Série, Tome xlv., 1878.

On the Instability of Jets. *By* Lord Rayleigh, F.R.S.

[*Read November 14th*, 1878.]*

Many, it may even be said, most of the still unexplained phenomena of Acoustics are connected with the instability of jets of fluid. For this instability there are two causes; the first is operative in the case of jets of heavy liquids, *e.g.*, water, projected into air (whose relative density is negligible), and has been investigated by Plateau in his admirable researches on the figures of a liquid mass, withdrawn from the action of gravity. It consists in the operation of the capillary force, whose effect is to render the infinite cylinder an unstable form of equilibrium, and to favour its disintegration into detached masses whose aggregate surface is less than that of the cylinder. The other cause of instability, which is operative even when the jet and its environment are of the same material, is of a more dynamical character.

With respect to instability due to capillary force, the principal problem is the determination, as far as possible, of the mode of disintegration of an infinite cylinder, and in particular of the number of masses into which a given length of cylinder may be expected to distribute itself. It must, however, be observed that this problem is not so definite as Plateau seems to think it; the mode of falling away from unstable equilibrium necessarily depends upon the peculiarities of the small displacements to which a system is subjected, and without which the position of equilibrium, however unstable, could not be departed from. Nevertheless, in practice, the latitude is not very great, because some kinds of disturbance produce their effect much more rapidly than others. In fact, if the various disturbances be represented initially by $\alpha_1, \alpha_2, \alpha_3, \ldots$, and after a time t by $\alpha_1 e^{q_1 t}$, $\alpha_2 e^{q_2 t}$, $\alpha_3 e^{q_3 t}, \ldots$, the (positive) quantities q_1, q_2, q_3, &c., being in descending order of magnitude, it is easy to see that, when $\alpha_1, \alpha_2, \ldots$ are small enough, the first kind necessarily acquires the preponderance. For example, at time t the ratio of the second kind to the first is $\frac{\alpha_2}{\alpha_1} e^{-(q_1 - q_2)t}$, which, independently of the value of $\alpha_2 : \alpha_1$, can be made as small as we please by taking t great enough. But, in order to allow the application of the analytical expressions for so extended a time, it is generally necessary to suppose the whole amount of disturbance to be originally extremely small.†

* Additions made since the reading of the paper are enclosed in square brackets. *April*, 1879.

† Some of the theorems given in the Proceedings for June 1873 (Theory of Sound, §§ 88, 89), for the periods of vibrations about a configuration of stable equilibrium, are applicable, *mutatis mutandis*, to the times of falling away from unstable equilibrium when various types of displacement are considered. For example, the application of a constraint could never diminish the shortest time previously possible.

Let us, then, taking the axis of z along the axis of the cylinder, suppose that at time t the surface of the cylinder is of the form

$$r = a + \alpha \cos \kappa z \quad \ldots\ldots (1),$$

where α is a small quantity variable with the time, and $\kappa = 2\pi\lambda^{-1}$, λ being the *wave-length* of the original disturbance. The information that we require will be readily obtained by Lagrange's method, when we have calculated expressions for the potential and kinetic energies of the motion represented by (1).

The potential energy due to the capillary forces is a question merely of the surface of the liquid. If we denote the surface corresponding (on the average) to the unit length along the axis by σ, we readily find

$$\sigma = 2\pi a + \tfrac{1}{2}\pi a \kappa^2 \alpha^2 \quad \ldots\ldots (2).$$

In this, however, we have to substitute for a (which is not strictly constant) its value obtained from the condition that S, the volume enclosed per unit of length, is given. We have

$$S = \pi a^2 + \tfrac{1}{2}\pi \alpha^2 \quad \ldots\ldots (3),$$

whence

$$a = \sqrt{\left(\frac{S}{\pi}\right) \cdot \left(1 - \tfrac{1}{4}\frac{\pi\alpha^2}{S}\right)} \quad \ldots\ldots (4).$$

Using this in (2), we get with sufficient approximation

$$\sigma = 2\sqrt{(\pi S)} + \frac{\pi\alpha^2}{2a}(\kappa^2 a^2 - 1) \quad \ldots\ldots (5);$$

or, if σ_0 be the value of σ for the undisturbed condition,

$$\sigma - \sigma_0 = \frac{\pi\alpha^2}{2a}(\kappa^2 a^2 - 1) \quad \ldots\ldots (6).$$

From this we infer that, if $\kappa a > 1$, the surface is greater after displacement than before; so that, if $\lambda < 2\pi a$, the displacement is of such a character that with respect to it the system is stable. We are here concerned only with values of κa less than unity. If T_1 denote the cohesive tension, the potential energy V reckoned per unit of length from the position of equilibrium is

$$V = -T_1 \frac{\pi\alpha^2}{2a}(1 - \kappa^2 a^2) \quad \ldots\ldots (7).$$

We have now to calculate the kinetic energy of motion. It is easy to prove that the velocity potential is of the form

$$\phi = A J_0 (i\kappa r) \cos \kappa z \quad \ldots\ldots (8),$$

J_0 being the symbol of Bessel's functions of zero order, so that

$$J_0 (x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\ldots \quad \ldots\ldots (9).$$

The coefficient A is to be determined from the consideration that the outwards normal velocity at the surface of the cylinder is equal to $\dot{a}\cos\kappa z$. Hence

$$i\kappa A J_0'(i\kappa a) = \dot{a} \quad \ldots\ldots(10).$$

Denoting the density by ρ, we have for the kinetic energy the expression

$$T = \tfrac{1}{2}\rho \int 2\pi a \,.\, \phi \, \frac{d\phi}{dr_{(=a)}}\, dz;$$

or, if we reckon it in the same way as V per unit of length,

$$T = \tfrac{1}{2}\rho\pi a^2 \frac{J_0(i\kappa a)\,\dot{a}^2}{i\kappa a\, J_0'(i\kappa a)} \quad \ldots\ldots(11).$$

Thus, by Lagrange's method, if $a \propto e^{qt}$,

$$q^2 = \frac{T_1}{\rho a^3} \frac{(1-\kappa^2a^2)\,.\,i\kappa a\,.\,J_0'(i\kappa a)}{J_0(i\kappa a)} \quad \ldots\ldots(12),$$

which determines the law of falling away from equilibrium for a disturbance of wave-length λ. The solutions for the various values of λ and the corresponding energies are independent of one another; and thus, by Fourier's theorem, it is possible to express the condition of the system at time t, after the communication of any infinitely small disturbances symmetrical about the axis. But what we are most concerned with at present is the value of q^2 as a function of κa, and especially the determination of that value of κa for which q^2 is a maximum. That such a maximum must exist is evident *a priori*. Writing x for κa, we have to examine the values of

$$\frac{(1-x^2)\,.\,ix\,.\,J_0'(ix)}{J_0(ix)} \quad \ldots\ldots(13).$$

Expanding in powers of x, we may write, for (13),

$$\tfrac{1}{2}x^2(1-x^2)\left\{1 - \frac{2}{2^3} + \frac{x^4}{2^4.3} - \frac{11x^6}{2^{10}.3} + \frac{19x^8}{2^{11}.3.5} + \ldots\right\} \quad \ldots\ldots(14).$$

or

$$\tfrac{1}{2}\left\{x^2 - \tfrac{2}{3}x^4 + \frac{7}{2^4.3}x^6 - \frac{25}{2^{10}}x^8 + \frac{91}{2^{11}.3.5}x^{10} + \ldots\right\} \quad \ldots\ldots(15).$$

Hence, to find the maximum, we obtain by differentiation

$$1 - \tfrac{2}{4}x^2 + \frac{7}{2^4}x^4 - \frac{100}{2^{10}}x^6 + \frac{91}{2^{11}.3}x^8 + \ldots = 0 \quad \ldots\ldots(16).$$

If the last two terms be neglected, the quadratic gives $x^2 = \cdot4914$. If this value be substituted in the small terms, the equation becomes

$$\cdot98928 - \tfrac{2}{4}x^2 + \tfrac{7}{16}x^4 = 0,$$

whence

$$x^2 = \cdot4858 \quad \ldots\ldots(17).$$

The corresponding value of λ is given by

$$\lambda = 4{\cdot}508 \times 2a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

which gives accordingly the ratio of wave-length to diameter for the kind of disturbance which leads most rapidly to the disintegration of the cylindrical mass. The corresponding number obtained by Plateau from some experiments by Savart is 4·38, but this estimate involves a knowledge of the coefficient of contraction of a jet escaping through a small hole in a thin plate, and is probably liable to a greater error than its deviation from 4·51.

The following table exhibits the relationship between x^2 or $\kappa^2 a^2$ and the square root of expression (13) to which q is proportional:—

x^2		x^2	
·05	·1536	·4	·3382
·1	·2108	·5	·3432
·2	·2794	·6	·3344
·3	·3182	·8	·2701
		·9	·2015

In the cases just considered, the cause of instability is statical, and the phenomena are independent of the general translatory motion of the jet; but the other kind of instability has its origin in this very translatory motion. In his work on the discontinuous movements of fluids, Helmholtz* remarks upon the instability of surfaces separating portions of fluid which move discontinuously, and Sir W. Thomson,† in treating of the influence of wind on waves in water, supposed frictionless, has shewn under what conditions a level surface of water is rendered unstable. In the following investigations the method of Thomson's paper is applied to determine the law of falling away from unstable equilibrium in some of the simpler cases of a plane or cylindrical surface of separation.

Let us suppose that the equilibrium position of the surface of separation is represented by $z = 0$, and that on the positive and negative sides of it the velocities of the fluid are parallel to the axis of x, and of magnitudes V and V' respectively. In the absence of friction, the motion consequent upon any deformation of the surface of separation is determinate, in virtue of a well known hydrodynamical law. By Fourier's theorem, any displacement in two dimensions can be resolved into component displacements of the undulatory type, and the effect of any two undulatory displacements may be considered separately. We might, therefore, take as the initial equation of the

* Phil. Mag., Vol. xxxvi. 1868. † Phil. Mag., Nov. 1871.

surface of separation $h = H\cos\kappa x$, in which h denotes the elevation at any point, λ the wave-length of the disturbance, and $\kappa = 2\pi\lambda^{-1}$. But, as in almost all such cases, it is more convenient to use complex expressions, from which the imaginary parts are finally rejected. We will therefore assume

$$h = H e^{int} e^{i\kappa x} \quad \ldots\ldots (19);$$

and the principal question which we have to consider is the dependence of n upon κ or λ.

For the velocity potential of the fluid on the positive side, we may take

$$\phi = A e^{int} e^{i\kappa x} e^{-\kappa z} + Vx \quad \ldots\ldots (20),$$

in which A is to be determined by equating the value of the normal velocity at the surface of separation with that obtained from (21). Thus (the positive direction of z being downwards)

$$-\frac{d\phi}{dz}_{(z=0)} = \kappa A e^{int} e^{i\kappa x} = \frac{dh}{dt} + V\frac{dh}{dx} = (in + i\kappa V) H e^{int} e^{i\kappa x},$$

whence $$A = i\kappa^{-1}(n+\kappa V) H \quad \ldots\ldots (22);$$

so that $$\phi = i\kappa^{-1}(n+\kappa V) H e^{int} e^{i\kappa x} e^{-\kappa z} + Vx \quad \ldots\ldots (23).$$

Similarly, for the fluid on the negative side,

$$\phi' = -i\kappa^{-1}(n+\kappa V') H e^{int} e^{i\kappa x} e^{\kappa z} + V'x \quad \ldots\ldots (24).$$

We have now to satisfy the condition of the equality of pressures. If σ denote the density, the hydrodynamical equation of pressure for the first fluid is

$$p = C - \sigma\frac{d\phi}{dt} - \tfrac{1}{2}\sigma U^2 \quad \ldots\ldots (25);$$

and approximately, when $z = 0$,

$$\tfrac{1}{2}U^2 = \tfrac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \tfrac{1}{2}\left(\frac{d\phi}{dy}\right)^2 = \tfrac{1}{2}V^2 - V(n+\kappa V) H e^{int} e^{i\kappa x} \quad \ldots\ldots (26).$$

In like manner, $$p' = C' - \sigma'\frac{d\phi'}{dt} - \tfrac{1}{2}\sigma' U'^2 \quad \ldots\ldots (27),$$

where $$\tfrac{1}{2}U'^2 = \tfrac{1}{2}V'^2 + V'(n+\kappa V') H e^{int} e^{i\kappa x} \quad \ldots\ldots (28).$$

Hence $$\sigma(n+\kappa V)^2 + \sigma'(n+\kappa V')^2 = 0 \quad \ldots\ldots (29),$$

which is the equation by which n and κ are connected.

The simplest case of (29) occurs when $\sigma' = \sigma$, and $V' = -V$, so that the equilibrium motions of the portions of fluid are equal and opposite. We have then

$$n^2 + \kappa^2 V^2 = 0 \quad \ldots\ldots (30);$$

and for the elevations

$$h = H e^{\mp\kappa Vt} \begin{matrix}\cos\kappa x \\ \sin\kappa x\end{matrix} \quad \ldots\ldots (31).$$

If initially $\frac{dh}{dt} = 0$, we get

$$h = H \cosh \kappa V t \begin{matrix} \cos \kappa x \\ \sin \kappa x \end{matrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32) ;$$

indicating that the waves on the surface of separation are *stationary*, and increase in amplitude with the time according to the law of the hyperbolic cosine. By (31), Fourier's theorem allows us to express the consequences of arbitrary initial values of h and $\frac{dh}{dt}$, within the limits of time imposed by our methods of approximation.

Next, let us suppose that $\sigma' = \sigma$, $V' = 0$. We get from (29)

$$n = \tfrac{1}{2}\kappa V(-1 \pm i) \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

whence $$h = He^{\mp \frac{1}{2}\kappa V t} e^{i(-\frac{1}{2}\kappa V t + \kappa x)},$$

of which the real part is

$$h = He^{\mp \kappa V t} \cos \kappa (\tfrac{1}{2} V t - x) \ldots\ldots\ldots\ldots\ldots (34).$$

In (34) an arbitrary constant may of course be added to x. It appears that the waves travel in the same direction as the stream, and with *one-half* its velocity. [In the case of the positive exponent, the rapidity with which the amplitude increases is very great. Since $\kappa = 2\pi\lambda^{-1}$, the amplitude is multiplied by e^{π}, or about 23, in the time occupied by the stream in passing over a distance λ. If $\lambda = V\tau$, $e^{\frac{1}{2}\kappa V t} = e^{\pi \frac{t}{\tau}}$, independent of V.]

As a generalised form of (34), we may take

$$h = A \cosh (\tfrac{1}{2}\kappa V t) \cos \kappa (\tfrac{1}{2} V t - x) + B \sinh (\tfrac{1}{2}\kappa V t) \sin \kappa (\tfrac{1}{2} V t - x) \ldots (35),$$

which gives, when $t = 0$, $h = A \cos \kappa x$.

If $\frac{dh}{dt} = 0$ initially, $B = A$, by which the solution corresponding to a surface of separation initially displaced without velocity is determined.

If initially $h = 0$, and $\frac{dh}{dt}$ is finite, we have, as the appropriate form,

$$h = B \sinh (\tfrac{1}{2}\kappa V t) \cos \kappa (\tfrac{1}{2} V t - x) \ldots\ldots\ldots\ldots\ldots (36).$$

Again, suppose that $\sigma' = \sigma$, $V' = V$. In this case the roots of (29) are equal, but the general solution may be obtained by the usual method. From (29) we have

$$n = \tfrac{1}{2}\kappa [\pm i (V' - V) - (V' + V)] \ldots\ldots\ldots\ldots\ldots (39);$$

or, if we put $V' = V(1 + \alpha)$,

$$n = \tfrac{1}{2}\kappa V [\pm i\alpha - (2 + \alpha)] \ldots\ldots\ldots\ldots\ldots (40).$$

The corresponding solution for h is

$$h = e^{i\kappa x} e^{\frac{1}{2}i(2+\alpha)\kappa Vt}\left[Ae^{\frac{1}{2}\kappa Vt.\alpha} + Be^{-\frac{1}{2}\kappa Vt.\alpha}\right] \ldots\ldots\ldots\ldots (41),$$

where A and B are arbitrary constants.

Passing now to the limit when $\alpha = 0$, and taking new arbitrary constants, we get

$$h = e^{i\kappa x} e^{-i\kappa Vt}\ [A + Bt] \ldots\ldots\ldots\ldots\ldots\ldots (42);$$

or, in real quantities,

$$h = \begin{matrix}\cos\\ \sin\end{matrix} \kappa\,(Vt - x)\ [A + Bt] \ldots\ldots\ldots\ldots\ldots\ldots (42).$$

If initially $h = \cos\kappa x$, $\dfrac{dh}{dt} = 0$,

$$h = \cos\kappa\,(Vt - x) + \kappa Vt \sin\kappa\,(Vt - x) \ldots\ldots\ldots\ldots\ldots (43).$$

The peculiarity of this case is that previous to the displacement there is no real surface of separation at all. Its bearing upon the flapping of sails and flags will be evident.

The proportionality to $V^{-1}\lambda$ of the time of falling away from equilibrium follows from the principle of dynamical similarity, as there is no linear element but λ.

When $V' = V$, the solution is the same, whether $\sigma' = \sigma$ or not. For example, (31), (32), (33) are applicable when $\sigma' = 0$.

In general, the solution of (29) is

$$\frac{n}{\kappa} = -\frac{\sigma V + \sigma' V' \pm i\sqrt{(\sigma\sigma')\,.\,(V - V')}}{\sigma + \sigma'} \ldots\ldots\ldots\ldots (44).$$

If $\sigma V + \sigma' V' = 0$, n is a pure imaginary, and the waves are stationary.

We will now suppose that the two portions of fluid are limited by rigid walls whose equations are respectively $z = l$, $z = -l'$. Then, corresponding to $h = He^{int}e^{i\kappa x}$, we get for the velocity potentials, in place of (23), (24),

$$\phi = \ i\kappa^{-1}(n + \kappa V)\, H\, \frac{\cosh\kappa\,(z - l)}{\sinh\kappa l}\, e^{int}e^{i\kappa x} + Vx \ldots\ldots (45),$$

$$\phi' = -i\kappa^{-1}(n + \kappa V')\, H\, \frac{\cosh\kappa\,(z + l')}{\sinh\kappa l}\, e^{int}e^{i\kappa x} + V'x \ldots\ldots (46),$$

and, in place of (29),

$$\sigma\,(n + \kappa V)^2 \coth\kappa l + \sigma'\,(n + \kappa V')^2 \coth\kappa l' = 0 \ldots\ldots (47).$$

If $l' = l$, the result is the same as if l and l' were both infinite.

If l' be infinite, $\coth\kappa l' = 1$; (47) may then be applied to a jet of width $2l$, symmetrical and symmetrically displaced with respect to the line $z = l$, and moving with velocity V in an infinite mass whose velocity is V'. If $V' = 0$, $\sigma' = \sigma$, so that the jet is of the same density as its stationary environment, (47) becomes

$$(n + \kappa V)^2 \coth\kappa l + n^2 = 0 \ldots\ldots\ldots\ldots (48),$$

of which the solution is

$$n = \kappa V \frac{-1 \pm i\sqrt{(\tanh \kappa l)}}{1+\tanh \kappa l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (49),$$

a generalisation of (33).

Thus $$h = He^{\pm \mu \kappa V t} \cos \kappa \left[\frac{Vt}{1+\tanh \kappa l} - x \right] \quad \ldots\ldots\ldots\ldots (50),$$

where $$\mu = \frac{\sqrt{(\tanh \kappa l)}}{1+\tanh \kappa l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (51).$$

When κl is very small, we may take in place of (50)

$$h = He^{\pm \sqrt{(\kappa l)}\, \kappa V t} \cos \kappa (Vt - x) \ldots\ldots\ldots\ldots (52).$$

We see, from (52), that when l is small the time of falling away from equilibrium is increased.

If the condition to be satisfied at $z = l$ be $\phi = 0$, in place of $\frac{d\phi}{dz} = 0$, the value of ϕ is

$$\phi = -i\kappa^{-1} (n+\kappa V) H \frac{\sinh \kappa (z-l)}{\cosh \kappa l} e^{int} e^{i\kappa x} + Vx \ldots\ldots (53);$$

so that, if, as before, $\frac{d\phi'}{dz} = 0$ when $z = -l'$,

$$\sigma (n+\kappa V)^2 \tanh \kappa l + \sigma' (n+\kappa V')^2 \coth \kappa l' = 0 \quad \ldots\ldots\ldots (54).$$

If $l' = \infty$, $\sigma' = \sigma$, $V' = 0$,

$$(n+\kappa V)^2 \tanh \kappa l + n^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (55).$$

This is applicable to a jet of width $2l$, moving in still fluid with velocity V, and displaced in such a manner that the sinuosities of its two surfaces are parallel.

When κl is small, we have, approximately,

$$h = He^{\pm \sqrt{(\kappa l)} \cdot \kappa V t} \cos \kappa (\kappa l \,.\, Vt - x) \ldots\ldots\ldots\ldots (56).$$

By a combination of the solutions represented by (52), (56), we may determine the consequences of any displacements (in two dimensions) of the two surfaces of a thin jet moving with velocity V in still fluid of its own density.

[These solutions may be extended to cases where the surface of separation is not plane, provided that the velocities of the fluids be constant (V, V') along it. Let us suppose that ϕ, ψ are the velocity potential and stream functions for the steady motion of the first fluid, and that the surface of separation corresponds to $\psi = \psi_0$. At $\psi = \psi_1$ let there be a rigid barrier, which of course has no influence upon the steady motion. Then, if the elevation at any point s, measured along the surface of separation, be given by

$$h = He^{i\kappa s} e^{int},$$

the velocity-potential of the disturbed motion is

$$\phi+\delta\phi=\phi-i\kappa^{-1}H(n+\kappa V)\ \frac{\cosh\kappa V^{-1}(\psi-\psi_1)}{\sinh\kappa V^{-1}\psi_1}\ e^{int}e^{i\kappa V^{-1}\phi}.$$

If l be the width of a uniform stream of velocity V, whose whole amount is equal to that of the stream between $\psi=\psi_0$ and $\psi=\psi_1$, and if dashed letters denote the corresponding quantities for the second fluid, we get finally for the equation in n

$$\sigma\coth\kappa l\,(n+\kappa V)^2+\sigma'\coth\kappa l'\,(n+\kappa V')^2=0,$$

which is the same form as (47).]

We will now pass to the consideration of cylindrical surfaces of separation, limiting ourselves for simplicity to the case of disturbances symmetrical about the axis (x). If h denote the increment of distance of any point on the surface from the axis, we may take, as before,

$$h=He^{int}e^{i\kappa x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (57),$$

and the corresponding value of the velocity potential for the fluid inside the cylinder is

$$\phi=AJ_0(i\kappa r)\,e^{int}e^{i\kappa x}+Vx \quad \ldots\ldots\ldots\ldots\ldots\ (58),$$

in which A is to be determined by the condition relating to the normal velocity at the surface ($r=a$). Thus

$$\phi=\kappa^{-1}(n+\kappa V)\,H\frac{J_0(i\kappa r)}{J_0'(i\kappa a)}e^{int}e^{i\kappa x}+Vx \quad \ldots\ldots\ldots\ (59).$$

For the motion of the fluid outside the cylinder, we have, in the first place, the general form

$$\phi'=\ C(-r)^{-\frac{1}{2}}e^{\kappa r}\left\{1-\frac{1}{1.(-8\kappa r)}+\frac{1^2.3^2}{1.2.(-8\kappa r)^2}-\ldots\right\}$$
$$+D(-r)^{-\frac{1}{2}}e^{-\kappa r}\left\{1+\frac{1}{1.(-8\kappa r)}+\frac{1^2.3^2}{1.2.(-8\kappa r)^2}+\ldots\right\},$$

in which, however, by the condition at infinity, we are to put $C=0$. Writing for brevity

$$(-r)^{-\frac{1}{2}}e^{-\kappa r}\left\{1+\frac{1}{1.(-8\kappa r)}+\frac{1^2.3^2}{1.2.(-8\kappa r)^2}+\ldots\right\}=\chi(r)\ldots(60),$$

we have accordingly

$$\phi'=B\chi(r)\,e^{int}e^{i\kappa x}+V'x \quad \ldots\ldots\ldots\ldots\ (61);$$

or, on determining the value of B,

$$\phi'=i(n+\kappa V')\,H\frac{\chi(r)}{\chi'(a)}e^{int}e^{i\kappa x}+V'x \quad \ldots\ldots\ldots\ldots\ (62).$$

In the same manner as for plane surfaces, the condition of equality of pressures now gives

$$\sigma i\kappa^{-1}(n+\kappa V)^2\frac{J_0(i\kappa a)}{J_0'(i\kappa a)}+\sigma'(n+\kappa V')^2\frac{\chi(a)}{\chi'(a)}=0 \quad \ldots\ldots\ (63),$$

as the quadratic by which n is determined.

When κa is small, we may employ approximate values of the functions J_0 and χ.

Thus
$$\frac{J_0(i\kappa a)}{J_0'(i\kappa a)} = -\frac{2}{i\kappa a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (64).$$

For χ we have ("Stokes on Pendulums," *Camb. Trans.*, Vol. ix.)

$$\chi = \left(\log\left(\tfrac{1}{8}\kappa r\right) - \pi^{-\frac{1}{2}}\Gamma'\left(\tfrac{1}{2}\right)\right)\left(1 + \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2 . 4^2} + \ldots\right)$$
$$- \left(\frac{\kappa^2 r^2}{2^2} S_1 + \frac{\kappa^4 r^4}{2^2 . 4^2} S_2 + \ldots\right) \ldots\ldots (65),$$

where
$$S_n = 1^{-1} + 2^{-1} + 3^{-1} + \ldots + n^{-1} \ldots\ldots\ldots\ldots\ldots\ldots (66).$$

When κa is small,

$$\chi(a) = \log\left(\tfrac{1}{8}\kappa a\right) - \pi^{-\frac{1}{2}}\Gamma'\left(\tfrac{1}{2}\right),$$
$$\chi'(a) = a^{-1},$$

so that
$$\frac{\chi(a)}{\chi'(a)} = a\left\{\log\left(\tfrac{1}{8}\kappa a\right) - \pi^{-\frac{1}{2}}\Gamma'\left(\tfrac{1}{2}\right)\right\} \ldots\ldots\ldots\ldots\ldots (67).$$

If we suppose that $\sigma' = \sigma$, $V' = 0$, (63) becomes, for small values of κa,
$$(n + \kappa V)^2 + \kappa^2 a^2 \left\{\log\frac{8}{\kappa a} + \pi^{-\frac{1}{2}}\Gamma'\left(\tfrac{1}{2}\right)\right\} n^2 = 0 \quad \ldots\ldots\ldots (68).$$

Writing μ^2 for
$$\kappa^2 a^2 \left\{\log\frac{8}{\kappa a} + \pi^{-\frac{1}{2}}\Gamma'\left(\tfrac{1}{2}\right)\right\},$$

we get
$$n = \kappa V \frac{-1 \pm i\mu}{1 + \mu^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (69),$$

whence approximately
$$h = He^{\pm\mu\kappa V t}\cos\kappa(Vt - x) \quad \ldots\ldots\ldots\ldots\ldots (70).$$

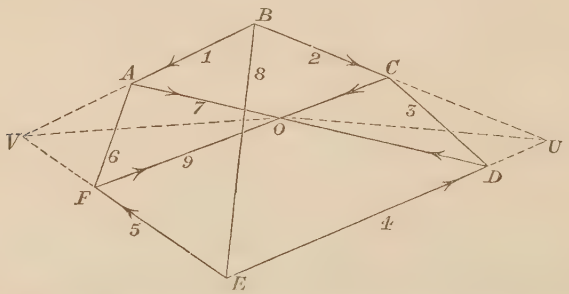
On Self-Strained Frames of Six Joints. *By* Prof. Crofton, F.R.S.

[*Read Nov. 14th*, 1878.]

In a frame of n joints, it is known that there must be $2n-3$ bars uniting these joints, in order to render the frame rigid, or *stiffen* it. If such a frame be subjected to the action of n given equilibrating forces, acting at the joints, and we consider the problem of determining the stresses in the $2n-3$ bars, we find that the equilibrium of the forces at each joint requires two conditions, or equations: giving in all $2n$ equations to be satisfied by the given forces and their directions, and the $2n-3$ unknown stresses. If these $2n-3$ unknowns were

eliminated from the $2n$ equations, we should find 3 equations to be satisfied by the given forces and their directions, which must be equivalent to the three conditions necessary in order that these forces should hold the frame in equilibrium, and therefore are already given by the conditions of the problem. There are, therefore, only $2n-3$ independent equations to determine the $2n-3$ stresses. It would appear, then, that we always have exactly the number of equations required, and that the problem is always a determinate one. In general it is so, and always so in the case of frames of 3, 4, or 5 joints, as is easy to verify (excepting a case of 5 joints, 3 of which are *in directum*). But when we come to frames of 6 joints (or any higher number), just stiffened by the proper number of bars, there are cases when the frame may be *self-strained*, that is, in a state of internal stress (of various degrees of intensity) independently of any external forces. This shows us that in these cases the problem above cannot be a determinate one, as we had supposed; because, whatever be the internal stresses in the frame when acted on by a given system of external forces, we can always superpose upon them a new system of stresses, due to any self-strained state of the frame, and we shall thus have a new state of internal equilibrium, under the action of the same external forces. We see thus that the $2n-3$ equations above referred to cannot in these cases be all independent. I do not propose to consider the analytical question, but only to determine the conditions that a frame of 6 joints, just stiffened by means of 9 bars, shall be capable of internal stress, without the action of external forces.

I. That a hexagonal frame $ABCDEF$, stiffened by means of diagonal bars AD, BE, CF, shall be capable of being self-strained, it is necessary and sufficient *that the* 6 *joints lie on a conic.*



For, suppose the frame in a state of stress, the 6 sides in thrust, and the three diagonals in tension; then any side, as AF, is in equilibrium under the four stresses in the bars 1, 5, 7, 9, as marked in the figure. Hence, producing 1, 5 to meet in V, we see that the resultant of the stresses in 7 and 9 passes through V, that is, it acts in OV. In the

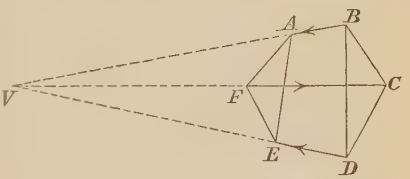
same way, the resultant of the stresses in 7 and 9, as acting on the bar CD, must act in OU. But these two directions are identical; consequently the three points V, O, U lie on a straight line; and this proves that $ABCDEF$ lie on a conic—being one of the forms of Pascal's theorem.

This condition is also *sufficient*. For, assuming any tension for the bar 7, T_7 suppose: we can find a value for T_9 such as to make the resultant of the two act in OV; we can then determine T_1T_5 so as to make their resultant equal and opposite to that of T_7T_9; consequently the bar 6 would be in equilibrium. We can also determine T_2T_4, as acting on the bar 3, so as to have a resultant equal and opposite to that of T_7T_9: so that CD would be in equilibrium. Also, considering the equilibrium of the bar DE, the resultant of the stresses in 1 and 5 is equal and opposite to that of the stresses in 2 and 4; so that BE is in equilibrium. Consequently, there is equilibrium at every joint. The stresses in all the other bars follow from the original value assumed for T_7, and all vary proportionally to that value.

The same proof applies if the hexagon be re-entrant.

II. There is only one other case when a frame of 6 joints and 9 bars can be self-strained.

When the hexagon is stiffened by the bars AE, BD, FC: Here the rigid triangle AFE is in equilibrium under the three stresses in BA, DE, FC as marked in the figure; consequently, these three directions produced must meet in a point V. Any values for these three stresses which satisfy the conditions for equilibrium at V, will then keep the triangle AFE in equilibrium; and also the triangle BCD; hence the frame will be in equilibrium. The only condition required being *that the lines* BA, DE, CF *shall meet in one point.*



It is easy to show that every possible case of a self-strained frame of 6 joints is reducible to one or other of the above cases—omitting cases where 3 or more joints lie in a straight line—it being obviously necessary that it be stiffened in such a manner that 3 bars meet at every joint.*

* Thus, suppose the joint A joined with B, C, D. There are two cases:—(1.) If two of the latter, as B, C, are joined, then we shall easily see that DEF must also be joined; so that we have the seven bars AB, AC, BC, DE, DF, EF, AD; the two remaining ones must be BE, CF, or else BF, CE. This gives us Case II. above, of two rigid triangles with their vertices joined. (2.) When B, C, D are not joined—Here we must have AB, AC, AD, and BE, BF, CE, CF, DE, DF; so that we have the closed *hexagon* $ABEDFCA$, with the three *diagonals* AD, BF, EC; that is, Case I. above.

I have not yet attempted to extend the inquiry to frames of 7 or more joints. There are even cases where a frame of *less* than the number $2n-3$ of bars required for stiffness may be self-strained. Thus, a frame of the form of a regular octagon, with four diagonal bars crossing at the centre, would evidently be in equilibrium, if the diagonals be all in tension, of the same amount, and the sides in compression of the amount requisite to give equilibrium at each joint: yet the frame is not even rigid, and would require a thirteenth bar to render it so.*

The Calculus of Equivalent Statements (Third Paper).

By Hugh McColl, B.A.

[*Read November 14th*, 1878.]

The following formulæ are all either self-evident or easily verified and some of them will be found useful in abbreviating the operations of the calculus:—

(1) $1'=0, \quad 0'=1$;

(2) $1=1+a=1+a+b=1+a+b+c, \ \&c.$;

(3) $(ab+a'b')'=a'b+ab'$,
$(a'b+ab')'=ab+a'b'$;

(4) $a:a+b:a+b+c, \ \&c.$;

(5) $(a+A)(a+B)(a+C)\ldots=a+ABC\ldots$;

(6) $(a:b):a'+b$;

(7) $(a=b)=(a:b)(b:a)$;

(8) $(a=b):ab+a'b'$;

(9) $(A:a)(B:b)(C:c)\ldots:(ABC\ldots:abc\ldots)$;

(10) $(A:a)(B:b)(C:c)\ldots:(A+B+C+\ldots:a+b+c+\ldots)$;

(11) $(A:x)(B:x)(C:x)\ldots=(A+B+C+\ldots:x)$;

(12) $(x:A)(x:B)(x:C)\ldots=(x:ABC\ldots)$;

(13) $(A:x)+(B:x)+(C:x)+\ldots:(ABC\ldots:x)$;

(14) $(x:A)+(x:B)+(x:C)+\ldots:(x:A+B+C+\ldots)$.

Rule 19.—To test the equivalence of any two statements; say

$$f(a, b, c, \ldots) \text{ and } \varphi(a, b, c, \ldots).$$

---

* It is easy to show that the only condition necessary for any octagonal frame, with 4 diagonal bars joining opposite vertices, to be capable of internal stress, is that the 4 intersections of each pair of opposite sides shall be *in directum*.

Let α denote the one, and β the other. If $\alpha = \beta$, then $\alpha\beta' + \alpha'\beta = 0$; and, conversely, if $\alpha\beta' + \alpha'\beta = 0$, then $\alpha = \beta$.

Symbolically, this rule may be briefly expressed thus:

$$(\alpha = \beta) = (\alpha\beta' + \alpha'\beta = 0).$$

RULE 20.—To discover the redundant terms of any indeterminate statement.

These redundant terms are easily detected by mere inspection when they imply (or are multiples of) *single* co-terms, as in the expression

$$\underline{\alpha'\beta\gamma} + \alpha'\gamma + \underline{\alpha\beta\gamma'} + \beta\gamma'.$$

But when they do not imply single co-terms, but the sum of two or more co-terms, they cannot generally be thus detected by inspection. They can always, however, be discovered by the following rule, which includes all cases :—

Any term of an indeterminate statement may be omitted as redundant when this term, multiplied by the denial of the sum of all its co-terms gives the product zero; and if the product is not zero, the term cannot be cancelled as redundant.

Take, for example, the indeterminate statement

$$CD' + C'D + B'C' + B'D'.$$

Beginning with the first term, we get

$$CD'(C'D + B'C' + B'D')' = BCD'.$$

Hence, the first term CD' cannot be cancelled as redundant.

Taking next the second term, we get

$$C'D(CD' + B'C' + B'D')' = BC'D.$$

This shews that the second term $C'D$ cannot be cancelled as redundant.

We next take the third term, and get

$$B'C'(CD' + C'D + B'D')' = 0.$$

The third term $B'C'$ may therefore be omitted as redundant.

Omitting the third term as redundant, we finally try the fourth term $B'D'$, and get $B'D'(CD' + C'D)' = B'C'D'$.

Thus the fourth term cannot be omitted as redundant, *if we omit the third term.* But, since $B'D'(CD' + C'D + B'C')'$ is also zero, the fourth term may be omitted as redundant, *if we retain the third term.*

We may therefore omit either the third or the fourth term, but not both.

Note.—The process for discovering redundant terms may be much abbreviated by means of Rule 22, which see further on.

Thus $CD'(C'D + B'C' + B'D')' = CD'(0 + 0 + B')' = BCD'$;

and, again, $B'D'(CD' + C'D + B'C')' = B'D'(C + 0 + C')' = 0$.

On the Reduction of Statements to their Primitive Form.

Def. 14.—An indeterminate statement is said to be in its *primitive form* when it and all its parts are free from redundant terms and redundant factors.

For example, $a+ab+m+m'n$ is reduced to its primitive form when we omit the redundant term ab, and out of the last term strike out the unnecessary factor m'. For $a = a+ab$, and $m+n = m+m'n$, so that the primitive form of the expression is $a+m+n$.

Rule 21.—To reduce an indeterminate statement to its primitive form:—

Develope the denial of the statement by the formula

$$(A+B+C+\ldots)' = A'B'C'\ldots \quad \text{(see Rule 4)};$$

multiply together the indeterminate factors which the application of the formula

$$(ABC\ldots)' = A'+B'+C'+\ldots \quad \text{(see Rule 9)}$$

will then give rise to; and omit the redundant terms in the product (see Rule 20). Then develope the denial of this product, and go through the same process as before. The result will be the primitive form of the original given statement.

Take, for example, the indeterminate statement

$$A+BC+A'B'D+A'C'D.$$

Let α denote this statement.

Then $$\alpha' = A'(BC)'(A'B'D)'(A'C'D)', \quad \text{by Rule 4.}$$

Applying Rule 9 to the bracket factors, we get

$$\alpha' = A'(B'+C')(\overline{A}+B+D')(\overline{A}+C+D').$$

The terms over-lined in the last two brackets may be cancelled since they are inconsistent with their co-factor A'. Hence

$$\begin{aligned}\alpha' &= A'(B'+C')(B+D')(C+D') \\ &= A'(B'+C')(D'+BC) \quad \text{(see formula 5)} \\ &= A'D'(B'+C').\end{aligned}$$

Hence, by Rules 9 and 4, we get

$$\alpha = A+D+BC.$$

The primitive form of the given statement is therefore

$$A+D+BC.$$

As another example, take the statement

$$AB'C'+ABD+A'B'D'+ABD'+A'B'D.$$

Let α denote this statement.

Then $$a' = A'B + AB'C,$$

omitting redundant terms.

Hence $$a = (A+B')(A'+B+C') = AB + AC' + A'B' + B'C'.$$

The second or fourth term (AC' or $B'C'$) may be omitted as redundant, but not both. The given statement has therefore *two* primitive forms, namely, $AB + A'B' + B'C'$ and $AB + AC' + A'B'$.

Note.—When the given indeterminate statement re-appears as the final result of the process, we may, of course, infer that it is already in its primitive form.

The Method of Unit and Zero Substitution.

Def. 15.—Statements of the form

$$A,\ A+B,\ AB\,(C+D),\ AB+CD+E,\ \&c.,$$

will be called *unconditional statements*, to distinguish them from implications (or conditional statements) such as $A : B$, $A+B : CD$, &c.

Rule 22.—Let $f(x)$ denote any unconditional statement involving x. Then $$xf(x) = xf(1), \text{ and } x'f(x) = x'f(0).$$

More generally, let $f(a, b, c, \ldots, \alpha, \beta, \gamma, \ldots)$

be any unconditional statement involving a, b, c, &c. Then

$$abc \ldots \alpha'\beta'\gamma' \ldots f(a, b, c, \ldots \alpha, \beta, \gamma, \ldots) = abc \ldots \alpha'\beta'\gamma' \ldots f(1, 1, 1, \ldots, 0, 0, 0, \ldots),$$

always substituting 1 for the repetition of an outside factor, and 0 for its denial.

Proof:—Any unconditional statement $f(x)$ may evidently, by multiplication and the application of Rules 4 and 9, be reduced to an indeterminate statement of the form $\alpha x + \beta x' + \gamma$. Let therefore

$$f(x) = \alpha x + \beta x' + \gamma.$$

Then, however complicated may be the expressions denoted by α, β, γ, it is evident that $xf(x) = x(\alpha+\gamma)$, and $x'f(x) = x'(\beta+\gamma)$;

that is, $$xf(x) = xf(1), \text{ and } x'f(x) = x'f(0).$$

By the same reasoning,

$$ab'f(a, b) = ab'f(1, b) = ab'f(1, 0);$$

and the same mode of proof may evidently be extended so as to prove the general formula

$$abc \ldots \alpha'\beta'\gamma' \ldots f(a, b, c, \ldots, \alpha, \beta, \gamma, \ldots) = abc \ldots \alpha'\beta'\gamma' \ldots f(1, 1, 1, \ldots, 0, 0, 0, \ldots).$$

Hence may be proved the formula (given in my second paper) for solving the implication $f(x) : \phi(x)$. Thus, from the given implication

we get $$f(x)\,\phi'(x) : 0.$$

Therefore $$x'f(x)\,\phi'(x) : 0;$$

that is, $$x'f(0)\,\phi'(0) : 0.$$

Hence $$f(0)\,\phi'(0) : x, \text{ by Rule 14.}$$

Also we have $$xf(x)\,\phi'(x) : 0,$$

that is $$xf(1)\,\phi'(1) : 0.$$

Hence $$f(1)\,\phi'(1) : x', \text{ by Rule 14.}$$

From these two formulæ we also get, by transposition and Rule 9,

$$x : f'(1)+\phi(1),$$
$$x' : f'(0)+\phi(0).$$

Rule 23.—Let $f(x)$ denote any complex statement (conditional or unconditional) involving x. Then

$$xf(x) : f(1), \text{ and } x'f(x) : f(0).$$

More generally, let $f(a, b, c, \ldots\ \alpha, \beta, \gamma, \ldots)$

denote any complex statement (conditional or unconditional) involving $a, b, c,$ &c. Then

$$abc\ldots\,\alpha'\beta'\gamma'\ldots f(a\ b, c, \ldots, \alpha, \beta, \gamma, \ldots) : f(1, 1, 1, \ldots, 0, 0, 0, \ldots),$$

substituting 1 for the repetition of any statement, and 0 for its denial.

For when we know a statement to be true, we may replace it by 1 in any co-factor of it, and the denial of it by 0. In the antecedent of the above implication, if we know the factors $a, b, c, \ldots, \alpha', \beta', \gamma', \ldots$ to be true, it follows that we may replace each of the true statements $a, b, c, \ldots$ by 1, and the denials $\alpha, \beta, \gamma, \ldots$ of the true statements $\alpha', \beta', \gamma', \ldots$ by 0, in the co-factor $f(a, b, c, \ldots, \alpha, \beta, \gamma, \ldots)$.

As a simple illustration, let $f(x, y)$ denote

$$ax+b : cy+dy'+e.$$

Then $$xy'f(x, y) : f(1, 0),$$

that is, $$xy'f(x, y) : (a+b : d+e).$$

In other words, if we know x to be true, y to be false, and the implication $ax+b : cy+dy'+e$ to be true, we may infer that the implication $a+b : d+e$ is also true.

Rule 24.—When we have solved any implications with respect to any statement which they may involve and its denial (say x and x'), we may omit in the antecedents any terms which imply (or are multiples of) any other terms (or sum of terms), *whether the implying and implied statements belong to the same antecedent or not.*

Take, for example, the implications

$$\overline{m}+\underline{ab}+\overline{a'c}+b+d : x,$$

$$\overline{m}+\underline{a'b'}+a'+\overline{ad}+e : x',$$

which we may consider as the solution with respect to x and x' of some given implications. The terms under-lined and over-lined may both be omitted. The terms under-lined, ab and $a'b'$, may be omitted as *redundant*, because they are multiples respectively of their co-terms b and a'; while the terms over-lined, m, $a'c$, and ad, may be omitted as *zero*, because they are multiples respectively of m, a', and d (which are not their co-terms), and therefore contain the inconsistent compound factor xx'.

Note.—When our object is not the solution with respect to x and x' of any implication (or implications), but to *eliminate* x; then, though we may still omit redundant terms, such as ab and $a'b'$ in the preceding example, we must not omit zero terms such as m, $a'c$, and ad; for the discovery of zero terms, independent of x, is the very object of the elimination.

I will now illustrate the utility of the preceding rules and principles by one or two examples.

Let there be four given implications (or conditional statements), namely—

$$ax' : c+dy \quad \ldots\ldots (1),$$

$$bx : c+dy+e \quad \ldots\ldots (2),$$

$$a'b' : x+de'+c \quad \ldots\ldots (3),$$

$$a+b+c : x+y \quad \ldots\ldots (4);$$

and from these four implications let it be required to deduce an implication containing neither x nor x', nor y nor y'.

Solving first with respect to x and x', we get

$$ac'(d'+y')+0+a'b'(d'+e)c'+(a+b+c)y' : x,$$

$$0+bc'(d'+y')e'+0+0 : x'.$$

Hence

$$abc'(d'+y')e'+bc'(d'+y')e'(a+b)y' : 0.$$

The factor $a+b$ in the second term may be omitted, since it is implied in its co-factor b (see Formula 4). Omitting this factor, we get

$$bc'(d'+y')e'(a+y') : 0.$$

We have now to solve this last implication with respect to y and y'. These solutions are *

$$bc'e' : y, \quad bc'd'e'a : y'.$$

Hence, finally,

$$abc'd'e' : 0.$$

This last implication constitutes the required solution; and from it all possible conclusions may be obtained by inspection (see Rule 14).

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* Observe that, when $y = 0$, y' must $= 1$, so that the factors $d'+y'$ and $a+y'$, being each unity (by Formula 2), may be omitted; and when $y = 1$, $y' = 0$, so that $d'+y' = d'$, and $a+y' = a$.

The problem may also be solved as follows:—

Let $f(x, y)$ denote the product of the four given implications from which we have to eliminate x and y. Then, omitting implications of the form $0 : a$ and $a : 1$, because they convey no information, we get, by application of Rule 23 and Formulæ 6 and 2 of this paper,

$$xy\,f(x, y) : b'+c+d+e,$$
$$x'y f(x, y) : (a'+c+d)(a+b+de'+c),$$
$$xy' f(x, y) : b'+c+e,$$
$$x'y' f(x, y) : (a'+c)(a+b+de'+c)\,a'b'c'.$$

Hence, since $xy+x'y+xy'+x'y' = 1$,

we get, by Formula 10 of this paper, adding antecedents to antecedents, and consequents to consequents, and omitting redundant terms,*

$$f(x, y) : b'+c+d+e+(a'+\underline{c}+\underline{d})(a+b+\underline{de'}+c).$$

The terms under-lined in the two brackets may be omitted, since they are multiples of c and d, the second and third terms before the brackets. Hence $f(x, y) : b'+c+d+e+a'(a+b)$;

that is, $f(x, y) : a'+b'+c+d+e,$

for $b'+a'b = a'+b'.$

Transposing, we get $abc'd'e' : f'(x, y).$

But, since $f(x, y)$ denotes the product of the premises, if all the premises be correct, $f'(x, y) : 0$. Hence, assuming the correctness of the premises, $abc'd'e' : 0.$

Let us next take an example from Boole's "Laws of Thought" (p. 106), and reproduced with a different solution in Jevons's "Pure Logic," p. 69.

Following Prof. Jevons's notation (as far as my method will permit), but with a slight difference in the interpretation of his symbols, let A, B, C, D, E respectively denote the five *statements*, that a certain substance is *wealth*, *transferable*, *limited in supply*, *productive of pleasure*, *preventive of pain*. The premises of the problem will then be expressed by the equation $A = BC(D+E)$;

and we are required to express C in terms of A, B, D, eliminating E.

Assuming the truth of the premises, and using Rule 23 and Formulæ 2 and 8 of this paper, we get

$$CE : (A = B) : AB+A'B',$$
$$CE' : (A = BD) : ABD+A'B'+A'D'.$$

---

* Observe that the fourth consequent is a multiple of the second, since $a'c$ implies (or is a multiple of) $a'+c+d$ (see Formula 4).

Hence, adding antecedent to antecedent, and consequent to consequent (see Formula 10), we get (since $E+E'=1$)

$$C : AB+A'B'+A'D' \quad \ldots\ldots (1),$$

omitting two redundant terms ABD and $A'B'$.

Again, from the given premises we get

$$C'E : (A=0) : A',$$
$$C'E' : (A=0) : A'.$$

Hence $$C' : A' \quad \ldots\ldots (2).$$

From (1) and (2) we get, by transposition,

$$\overline{AB'}+A'BD : C',$$
$$A : C.$$

The term AB' over-lined in the first of these two implications may be cancelled as zero,* since it is a multiple of the term A in the second implication (see Rule 24). Cancelling this term and transposing again, we get, for our final result,

$$A : C : A+B'+D'.$$

Another problem from Boole (see "Laws of Thought," page 146), when translated into the language of this calculus, is the following:—

From the premises expressed by the complex statement

$$\{A'C' : E(B'D+BD')\}(ADE' : BC+B'C')\{A(B+E)=C'D+CD'\},$$

it is required, (1) to express A in terms of B, C, D, eliminating E; and (2) to express B in terms of A, C, D.

Using Rule 23 and Formulæ 6 and 8 of this paper, we get

$$AE : C'D+CD',$$
$$AE' : (D'+BC+B'C')\{B(C'D+CD')+B'(CD+C'D')\}.$$

Adding antecedent to antecedent, and consequent to consequent (see Formula 10), and omitting the term $B(C'D+CD')$ in the double bracket, because it is a multiple of the first consequent $C'D+CD'$, we get $$A : C'D+CD'+B'(D'+\overline{BC}+B'C')(CD+C'D').$$

We may cancel the term over-lined because it is inconsistent with the outside factor B'.

Hence $$A : C'D+CD'+B'(C'D'+\underline{B'C'D'}),$$

that is, $$A : C'D+CD'+B'C'D'.$$

Hence, by transposition, $$CD+BC'D' : A'.$$

---

* From this zero term we may deduce the two implications $A : B$ and $B' : A'$.

Again, from our premises we get

$$A'E : (C+B'D+BD')(CD+C'D'),$$
$$A'E' : C(CD+C'D').$$

Now, since C implies $C+B'D+BD'$, by Formula 4, the consequent of the second implication implies (or is a multiple of) the consequent of the first implication, and may therefore be omitted as redundant when the two consequents are added together. Hence we get

$$A' : (C+B'D+BD')(CD+C'D'),$$

that is,

$$A' : CD+BC'D'.$$

The consequent of A' in this last implication was before shewn to be the antecedent of A'. Hence

$$A' = CD+BC'D'.$$

Therefore

$$A = C'D+CD'+B'C' = C'D+CD'+B'D'.$$

These two equivalents for A are the two primitive forms of

$$(CD+BC'D')',$$

the denial of the equivalent for A' (see Rule 21).

This will be found to agree with Boole's result when his expression for A is reduced to its primitive form (see Rule 21).

We are next required to express B in terms of A, C, D.

Now, in doing this, we might go back to the given premises of the problem and proceed as before; but it will be much shorter to avail ourselves of the results already obtained. If we had over-lined any *conditional zero terms* (*i.e.*, terms which are zero by virtue of, and not independently of, the given implications of the problem), we might, by equating these with zero, obtain expressions for B (*i.e.*, antecedents and consequents of B) in terms of C and D, independently of A, and then obtain the antecedents and consequents involving A by means of the equation

$$A' = CD+BC'D'.$$

But since no conditional zero terms have been thus over-lined, all that remains to be done is to solve the last equation with respect to B and B', thus:—

$$B : A'(CD+C'D')+A(C'D+CD'),$$
$$B' : A'(CD)+A(CD)'.$$

Transposing, we get (by mere inspection, if we employ Formula 3)

$$A(CD+C'D')+A'(C'D+CD') : B',$$
$$A(CD)+A'(CD)' : B.$$

Writing the antecedents of B' and B in full (without brackets), we get

$$A\bar{C}\bar{D}+AC'D'+\overline{A'C'D}+\overline{A'CD'} : B',$$
$$A\bar{C}\bar{D}+A'C'+A'D' : B.$$

Omitting the conditional zero terms over-lined (see Rule 24), and transposing in the first implication, we get

$$B : A' + C + D,$$
$$A'(C' + D') : B.$$

Boole's conclusions, when translated into this notation, are

$$B : A'(C'D' + CD) + A(C'D + CD'),$$
$$A'C'D' : B.$$

The first of these implications has been already obtained, and that its consequent is equivalent to the simpler consequent $A' + C + D$ may be proved by Rule 19; thus,—Putting α and β for the two consequents respectively, we get $\alpha\beta' + \alpha'\beta = A'C'D + A'CD' + ACD$. Now the three terms on the right-hand side of this equation have already been over-lined as zero. Hence $\alpha\beta' + \alpha'\beta$ vanishes, which shows that the statements denoted by α and β are equivalent. Similarly it may be shown that the two antecedents $A'(C' + D')$ and $A'C'D'$ are equivalent.

Let it be required, next, to express D in terms of A, B, C.

E having already been eliminated, we may, as before, avail ourselves of the equation

$$A' = CD + BC'D'$$

previously obtained. From this we readily get

$$D : A'C + AC',$$
$$D' : A'(BC') + A(BC')'.$$

Transposing, we get (see Formula 3)

$$AC + A'C' : D',$$
$$ABC' + A'B' + A'C : D.$$

This gives us Prof. Boole's conclusions $AC : D'$, $A'C' : D'$, $A'C : D$ (which, being independent of B, might also have been obtained directly by inspection of the conditional zero terms already over-lined in the antecedent of B'), and also two conclusions $ABC' : D$ and $A'B' : D$, involving B and not given in Boole's solution.

If we were required to express C in terms of A, B, D, we might, as before, start from the equation

$$A' = CD + BC'D'.$$

Thus

$$C : A'D + AD',$$
$$C' : A'(BD') + A(BD')';$$

and, by transposition,

$$AD + A'D' : C',$$
$$A'B' + A'D + ABD' : C.$$

This solution with respect to C is not given in Boole's "Laws of Thought."

This calculus of statements and implications may, I believe, be employed with advantage in investigating the causes of natural phenomena. Let it be required, for example, to ascertain the cause of some phenomenon X, and let the data of the problem (data which may be the results of observation and experiment, or deductions from previously known laws) be expressed symbolically in one complex statement, say

$$\phi\,(x, a, b, c, \ldots, u, v, \ldots),$$

in which x asserts the existence (or occurrence) of the phenomenon X, while $a, b, c, \ldots, u, v, \ldots$ assert the existence of certain accompanying circumstances, $A, B, C, \ldots, U, V, \ldots$ If we know, or have strong reasons for suspecting, that certain of the monomial (or single-letter) statements, say $u, v, \ldots$, have reference to merely accidental and not necessary concomitants of X, our first step should be to eliminate these from the data. Let the implication left after this elimination be

$$\psi\,(x, a, b, c, \ldots) : 0.$$

We must now solve this implication with respect to x and x'. Let the solution (after omitting zero and redundant terms, and reducing the antecedents of x and x' to their primitive form) be

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots : x,$$
$$\beta_1 + \beta_2 + \beta_3 + \ldots : x',$$

in which $\alpha_1, \alpha_2, \alpha_3, \ldots, \beta_1, \beta_2, \beta_3, \ldots$ are some functions of $a, b, c, \ldots$

If the statements $\alpha_1, \alpha_2, \alpha_3, \ldots$ have a common factor, the circumstance (or combination of circumstances), whose existence is asserted by this common factor, is probably the cause of X. This follows from the canon of inductive philosophy called the "Method of Agreement." When this common factor is an indeterminate statement, the circumstances (or combinations of circumstances), whose existence is asserted by the several terms of this indeterminate statement, may (provisionally, till a common factor of the terms is discovered) be regarded as separate and independent causes of X.

Let m, asserting the existence of the circumstance M, be a common factor of the statements $\alpha_1, \alpha_2, \alpha_3, \ldots$. If $\beta_1, \beta_2, \beta_3, \ldots$ have also the common factor m', asserting the *non*-existence of M, the probability that M is the cause of X will be greatly increased. This is in accordance with the inductive canon called the "Joint Method."

If, finally, any term in the antecedent of x be of the form qm, while some term in the antecedent of x' is of the form qm', the sole difference being that m is accented in the one term and unaccented in the other; then the inference that M is the cause of X will be decisively confirmed by the inductive "Method of Difference."

In the abstract of this paper, read at the Meeting of the Mathematical Society, on the 14th of last November, mention was made of some criticism of mine on Prof. Jevons's method of solving logical problems.

This criticism, I wish to say now, was founded on an insufficient acquaintance with his method,—an acquaintance derived solely from his "Elementary Lessons in Logic,"—and for this reason I now ask permission to retract it. Prof. Jevons has since kindly sent me a copy of his "Pure Logic," dated 1864, from which I find that there is a much closer resemblance between his method and mine than I had any idea of, and that he had anticipated me in several things of which I believed myself to be the first discoverer.

I am also indebted to the kindness of the Rev. Robert Harley for the loan of Boole's "Laws of Thought," in which, as in Jevons's "Pure Logic," in spite of a very different notation and mode of treatment, I find many points of resemblance to my method. My method, however, differs both from Prof. Boole's and Prof. Jevons's in three cardinal points, which a perusal of this paper will show to be so important as to necessitate an essentially different treatment of the whole subject. These three points are:—

(1) With me every single letter, as well as every combination of letters, always denotes a *statement*.

(2) I use a symbol (the symbol :) to denote that the statement following it is true provided the statement preceding it be true.

(3) I use a special symbol—namely, an accent—to express denial; and this accent, like the minus sign in ordinary algebra, may be made to affect a multinomial statement of any complexity.

On these three points of difference are founded some symmetrical rules on which depends the whole working of the calculus.

It may interest some of the readers of this paper to be told that this method owes its origin to a question in probability (No. 3440), proposed by Mr. Stephen Watson in the *Educational Times*. My solution of this question, with an introductory article, entitled "Probability Notation," was published in the *Educational Times* for August, 1871; and in this introductory article may be seen the germs of the present method. Shortly after this I gave up all mathematical investigations, and my thoughts did not again revert to the subject till two or three months before the appearance of my article on "Symbolical Language" in the *Educational Times* for July, 1877. This article was in the editor's hands before I noticed that this "Symbolical Language," as I called it, might also be employed, without change or modification, and with unerring certainty, in tracing to their last hiding-place the limits which often escape so mysteriously from the mathematician's grasp when he ventures to change the order of integration or the variables in a multiple integral. This was a pleasant and encouraging surprise,

and gave rise to hopes that the discovery of fresh fields of application would soon follow. I still looked upon the method, however, as an essentially mathematical one—grafted indeed on a logical stem, but destined to yield mathematical fruit, and mathematical fruit only. In the application of my method to examples, I was in the habit (when applying Rule 8) of using the symbol : for the word *implies*, writing

$$x_{m'n} : p\,(x_m - x_n) : p\,\{\phi\,(y, z)\} : \&c.$$

as a convenient abbreviation for

$$x_{m'n} = x_{m'n}\,p\,(x_m - x_n) = x_{m'n}\,p\,\{\phi\,(y, z)\} = \&c.;$$

but I did not use this symbol in anything that I sent for publication, as it did not seem to involve any important principle, and I was unwilling to complicate my method by the introduction of any symbol that might, without any serious inconvenience, be dispensed with. Afterwards, however, when I began to turn my attention to the more purely logical aspect of the method, I saw that very important advantages might be secured by employing this or some other convenient symbol to express logical inference, especially in the applications of the very useful principle of "contraposition"; and the result of my cogitations upon this point was my second paper printed in these "Proceedings."

December 12*th*, 1878.

C. W. MERRIFIELD, Esq., President, in the Chair.

Prof. W. Stanley Jevons, F.R.S., was elected a Member.

The following communications were made:—

"On a Kinematic Paradox:" Mr. H. Perigal, F.R.A.S.

"On Forms of Numbers determined by Continued Fractions:" Mr. S. Roberts, F.R.S.

"On a Graphic Construction of the Powers of a Linear Substitution:" Prince Camille de Polignac.

The following presents were made to the Society:—

"Thèses présentées à la Faculté des Sciences de Paris pour obtenir le grade de Docteur ès Sciences Mathématiques," par M. Halphen (1°, Sur les invariants différentiels; 2°, Propositions données par la Faculté); Paris, 1878: from the Author.

"Skelet af en elementær-geometrisk Keglesnitslære af H. G. Zeuthen" ("Tidsskrift for Mathematik," 1878); 1878: from the Author.

"Mémoire sur la Théorie des Coordonnées curvilignes et des Systèmes orthogonaux," par M. G. Darboux ("Annales scientifiques de l'École Normale supérieure"); Paris, 187 : from the Author.

"Educational Times," Dec., 1878.

Carte-de-visite Likenesses of MM. Darboux and Sophus Lie, and a photographic copy of a likeness of Abel.

On Forms of Numbers determined by Continued Fractions.

By Samuel Roberts, M.A., F.R.S.

[*Read December 12th*, 1878.]

1. In my paper "On the Decomposition of Certain Numbers into Sums of Two Square Integers by Continued Fractions,"* I made use of considerations which are evidently susceptible of wider application. A somewhat detailed treatment of a particular case is not unlikely to be useful in connection with a theory which, like that of binary quadratic forms, presents highly condensed and technically expressed results. I shall now give some further illustrations of the subject in a more general aspect.

The theory of Legendre relative to primes of the form $4m+1$ was greatly extended by Goepel in his inaugural dissertation "De æquationibus secundi gradus indeterminatis" (1835).†

The principal aim of the essay was to show that the decomposition of a prime of the form $8m+3$ or its double into a square and the double of a square, or of a prime of the form $8m+7$ or its double into the difference of a square and the double of a square, is obtainable by continued fractions in a manner strictly analogous to that in which the decomposition of a prime of the form $4m+1$ is effected.

Goepel showed, in fact, that such a decomposition can be effected of all numbers whose square roots, developed as continued fractions with unit numerators, present 2 as the denominator of a complete quotient. And, as Prof. H. J. S. Smith has remarked, the method employed is yet more general.

2. In the Report on the Theory of Numbers (Part V., "British Association Reports," 33rd Meeting), Professor Smith has given the more general results to which Goepel's conclusions point. It would be difficult to abridge the reasoning without confusion, and I only mention therefore some of the principal results which have a bearing on what I have to say. The section referred to is entitled "Application of Continued Fractions to the Theory of Quadratic Forms." (pp. 783–6.)

The Author sets out from the following theorem:—

"If (a, b, c), (a', b', c') are two primitive forms of the determinants D, D', whose joint invariant $aa'-2bb'+cc'$ is zero, and if m and m' are the greatest common divisors of a, $2b$, c; a', $2b'$, c'; m^2D' and m'^2D are respectively capable of primitive representation by the duplicates of (a, b, c) and (a', b', c')."

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* Proceedings, Vol. IX., p. 187, *et seq.*

† Republished in Crelle's Journal, Vol. XLV., pp. 1—13.

If, then, $\frac{p_n}{q_n}$ is a convergent to $\sqrt{A}$, so that $p_n^2 - Aq_n^2 = (-)^n D_n$, and $(q_n, -p_n, q_nA)$ is properly primitive and of the principal class, or improperly primitive and of the principal class of its order, we may suppose the latter form to be changed into $(\epsilon_s, -\delta_s, \epsilon_{s-1})$ by a substitution $\begin{pmatrix} p_s, & p_{s-1} \\ q_s, & q_{s-1} \end{pmatrix}$, and that we have

$$(-)^{n+1} D_n = \epsilon_s \epsilon_{s-1} - \delta_s^2, \quad \epsilon_{s-1} D_s - 2\delta_s J_s - \epsilon_s D_{s-1} = 0, \quad A = J_s^2 + D_s D_{s-1},$$

and ϵ_s or $\epsilon_{s-1} = 1$ if the form is properly primitive, ϵ_s or $\epsilon_{s-1} = 2$ if it is improperly primitive.

If n is odd, so that $(-)^{n+1} D_n = \Delta$, the following theorem is arrived at:—

"If $\frac{p_n}{q_n}$ is an inferior convergent to $\sqrt{A}$, and $\Delta = q_n^2 A - p_n^2$; when $(q_n, -p_n, q_nA)$ is of the principal class of forms of determinant $-\Delta$, A is of the form $X^2 + \Delta Y^2$, and Y is the denominator of a complete quotient in the development of $\sqrt{A}$; when $(q_n, -p_n, q_nA)$ is of the principal class of improperly primitive forms of determinant $-\Delta$, $2A$ is of the form $2X^2 + 2XY + \frac{\Delta+1}{2} Y^2$, and Y is the denominator of a complete quotient in the development of $\sqrt{A}$."

Similar results are noticed when $(q_n, -p_n, q_nA)$ is ambiguous and properly primitive, but not of the principal class; or ambiguous and improperly primitive, but not of the principal class of improperly primitive forms.*

In the case of n even, corresponding expressions are also given, and, in conclusion, some account is furnished of the speciality of the middle denominator of a period, particularly when it is 1 or 2.

3. The middle denominator in the periods of the development of $\sqrt{A}$ as a continued fraction has naturally some characteristics deserving of attention.

We may consider the set of denominators, commencing with the first denominator unity, and ending with the denominator immediately preceding the next denominator whose value is unity, and whose place is odd, as constituting the first period. Thus considered, the period has a middle term. When this term is odd, it is a factor of A; and when it is even, its half is a factor of A.

If, therefore, (t, u) is the least solution of $x^2 - Ay^2 = 1$, we shall have (except as hereinafter mentioned), for some values t', u' less than

* In this connection should be mentioned Goepel's theorem $\Delta > -\delta_s \delta_{s+1}$, δ_{s+1} being negative (p. 784, *loc. cit.*). The usefulness of this is easy to see, especially when Δ is small.

t, u, either $Mt'^2 - Nu'^2 = \pm 1, \quad MN = A$ ..... (a),

or $Mt'^2 - Nu'^2 = \pm 2, \quad MN = A$ ..... (b).

If A is the double of an odd number, so that $N = 2K$, both forms of equation hold good, and we have, from (b) for instance,

$$2Mt'^2 - Ku'^2 = \pm 1.$$

This is the case also whenever $A = MN$ is even, and $(M, 0, -N)$ is properly primitive and gives (b), or when (b) is of the form

$$2^m Pt'^2 - 2Qu'^2 = \pm 2,$$

and P, Q are odd, and t' is even.

It is necessary, on account of our original assumption, to except the use of $M = 1$, $t'^2 - Nu'^2 = 1$, $N = A$.

If $M = 1$ and $t'^2 - Au'^2 = -1$, the period of denominators possesses a double symmetry, and consists of two subordinate periods without middle terms.

4. After showing that the middle denominator, or the half of it, is a factor of A, and thus establishing that any number not a square can be separated into two factors M, N, so that

$$Mx^2 - Ny^2 = \pm 1 \quad \text{or} \quad Mx^2 - Ny^2 = \pm 2,$$

Legendre remarks as follows :—*

"Il faut d'ailleurs observer—1°, que pour un même nombre $A = MN$, il n'y aura jamais q'une manière de satisfaire à l'une de ces équations ; car il n'y a q'un quotient moyen qui résulte du développement de $\sqrt{A}$ en fraction continue."

The above reasoning seems to require at least some expansion. A denominator may be a factor of A, and yet not a middle denominator corresponding to a middle quotient ; viz., it may be the factor of A conjugate to the middle denominator, *e.g.*, in the case of $\sqrt{45}$. Or, again, the middle denominator may occur again in the period, and therefore not as a middle term, as in the case of $\sqrt{153}$. The conclusion that M or N, or the double of M or N, must be a middle denominator, is therefore not obvious, and requires further consideration.

5. I shall now show that, if the equation (a) is resoluble, and M is $< N$, *i.e.*, $< \sqrt{A}$, M is the middle denominator of the period arising from the development of $\sqrt{A}$ as a continued fraction.

In fact, the form $(M, 0, -N)$ or else $(N, 0, -M)$ is of the principal class, and the continued fraction representing $\frac{\sqrt{A}}{M}$ has the same denominator periods as that representing $\sqrt{A}$, beginning however with the

* Théorie des Nombres, Tome I., p. 69.

denominator M, which may be regarded as a middle term of denominator periods, and is consequently the middle term of the denominator period of $\sqrt{A}$. The middle term, then, if it is odd, or if it is even, and A is the double of an odd number; or if A is even, and $(M, 0, -N)$ in (b) is properly primitive, or if (b) is of the form

$$2^m Pt'^2 - 2Qu'^2 = \pm 2,$$

and P, Q are odd and t' is even, is $< \sqrt{A}$.

6. When (b) holds, but not (a), the middle denominator belonging to $\sqrt{A}$ is $2M$, M being $< N$. For, if we develope $\frac{\sqrt{A}}{M}$ as a continued fraction, and $\frac{p_{s-2}}{q_{s-2}}$, $\frac{p_{s-1}}{q_{s-1}}$ are consecutive convergents, with $\frac{\sqrt{A}+J_s}{D_s}$ for the next complete quotient, we have

$$\frac{\sqrt{A}}{M} = \frac{p_{s-1}(\sqrt{A}+J_s)+p_{s-2}D_s}{q_{s-1}(\sqrt{A}+J_s)+q_{s-2}D_s},$$

and thence

$$q_{s-1}J_s + q_{s-2}D_s = Mp_{s-1},$$
$$p_{s-1}J_s + p_{s-2}D_s = Nq_{s-1}.$$

When $D_s = 2$,

$$J_s + 2\frac{q_{s-2}}{q_{s-1}} = M\frac{p_{s-1}}{q_{s-1}}.$$

Let a be the greatest integer in $\sqrt{A}$. Since

$$(Mp_{s-1})^2 - Aq_{s-1}^2 = \pm 2M, \quad M\frac{p_{s-1}}{q_{s-1}} \text{ is } \overline{\underline{>}}\, a \overline{\underline{<}}\, a+1.$$

For, if we take the upper sign, $\frac{Mp_{s-1}}{q_{s-1}}$ is evidently $> a \overline{\underline{<}}\, a+1$. If the lower sign be taken, we have

$$(Mp_{s-1})^2 = Aq_{s-1}^2 - 2M \overline{\underline{>}}\, Aq_{s-1}^2 - 2Mp_{s-1},$$

which is positive for $A > 1$, since $2Mp_{s-1}$ is $< 2\sqrt{Aq_{s-1}^2}$.

Therefore

$$(Mp_{s-1}+1)^2 > Aq_{s-1}^2,$$
$$Mp_{s-1} < \sqrt{Aq_{s-1}^2},$$

so that

$$Mp_{s-1} \overline{\underline{>}}\, aq_{s-1} < (a+1)\, q_{s-1}.$$

Also $\frac{2q_{s-2}}{q_{s-1}}$ is < 2, and consequently, if A and a are of the same species, $J_s = a$, and if they are of different species $J_s = a-1$, because $A - J_s^2$ is even.

In developing $\frac{\sqrt{A}}{M}$, we shall arrive, then, either at a complete quotient $\frac{\sqrt{A}+a}{2}$, or a complete quotient $\frac{\sqrt{A}+a-1}{2}$, and the correspond-

ing partial quotients will be a or $a-1$. There is, therefore, only one form of equation (b), and $2M$ is the middle denominator belonging to $\sqrt{A}$ and may be $> \sqrt{A}$; *e.g.*, in the case of $\sqrt{91}$, and generally when $2M$ is $> N$ and $N > M$.

If M and N are both even, so that (b) is of the form

$$2^m P x^2 - 2Qy^2 = \pm 2,$$

where P and Q are odd and x^2 is also odd, $m > 1$, the middle denominator will be the least of the numbers $2^{m-1}P$ and 2^2Q. If x is even, and $Q < 2^{m-1}P$, Q will be the middle denominator, except that, if $Q = 1$, it may hold an odd place.

The numbers 1 and 2 can only appear as middle denominators in the development of $\frac{\sqrt{A}}{M}$. In fact, if μ_s is the partial quotient corresponding to J_s, D_s, we have $\mu_s D_s = J_s + J_{s-1}$. Hence, if $D_s = 2$, $\mu_s = a$ or $a-1$, and $J_s = J_{s-1}$; D_s is therefore a middle term.

The case of 1 as a denominator is similar.

7. If the number A is an odd power of an odd prime, or the double of an odd or even power of such a prime, its middle denominator is 1 or 2.* In particular, if A is an odd power of a prime of the form $4m+1$, or the double of an odd power of a prime of the form $8m+5$, the middle denominator is unity.

The last conclusion follows from the consideration that, if P is a prime of the form $8m+5$, we must have either

$$x^2 - 2Py^2 = -1 \quad \text{or} \quad x^2 - 2Py^2 = \pm 2 \ \ldots\ldots\ldots\ (1, 2),$$

or

$$2x^2 - Py^2 = \pm 1 \quad \text{or} \quad 2x^2 - Py^2 = \pm 2 \ \ldots\ldots\ldots\ (3, 4).$$

But (2) and (3) are equivalent, since, in (2), x is even. Also (4) gives y even, so that $x^2 - 2Py^2 = \pm 1$, and the upper sign is excluded because (x, y) may be taken less than the least solution of $x_1^2 - 2Py_1^2 = 1$. Hence we have the alternatives $x^2 - 2Py^2 = -1$ or $2x^2 - Py^2 = \pm 1$. The latter form is found to be impossible by exclusion of multiples of 8, and the former equation must hold.

If A is an odd power of a prime of the form $4m+3$, or the double of such a number, the middle denominator is 2. This is also the case when A is the double of an even power of such a prime.

In the cases of the double of an odd or even power of a prime of the form $8m+1$, and of the double of an even power of a prime of the form $8m+5$, the middle denominator may be 1 or 2.

8. If $Mx^2 - Ny^2 = \pm 1$ is resoluble, we may write

$$(\sqrt{M}.x - \sqrt{N}.y)^{2m+1} = \sqrt{M}.X - \sqrt{N}.Y,$$

$$MX^2 - NY^2 = \pm 1.$$

* This follows immediately from Legendre's theorem mentioned in § 4.

Hence, if $2m+1$ contains any odd factor of M or N, X or Y will contain that factor. It follows that, if we augment the index of any power of an odd prime contained in M or N as a factor by an even number, the equation remains resoluble.

In like manner, if the equation

$$Mx^2-Ny^2=\pm 2$$

is resoluble, we may write

$$\left(\sqrt{\frac{M}{2}}\cdot x-\sqrt{\frac{N}{2}}\cdot y\right)^{2m+1}=\sqrt{\frac{M}{2}}\cdot X-\sqrt{\frac{N}{2}}\,Y,$$

$$MX^2-NY^2=\pm 2.$$

And, therefore, if we augment the index of a power of an odd prime contained as a factor in M or N by an even number, the equation remains resoluble.

Hence, if M or $2M$ is the middle denominator in $\surd A$, and we augment the index of any odd prime factor of N by an even number, the middle denominator remains the same. But if we do the same thing with regard to factors of M, so that the augmented number becomes $> N$, the middle denominator will be N or $2N$. And generally, if we so augment the factors of both M and N, the middle denominator will be the least of the numbers so augmented, or the double of it. Thus, if P, P' are primes of the form $4m+3$, the development of $\surd P^{2p+1}.P'^{2p'+1}$ has the least of the numbers P^{2p+1}, $P'^{2p'+1}$ for the middle denominator. Again, $51=7^2+2$ gives $3^{2a+1}.17^{2b+1}=$ square and double of a square determined by continued fractions.

9. The identity

$$(au\alpha+b(u\beta+v\alpha)+cv\beta)^2-(b^2-ac)(u\beta-v\alpha)^2$$
$$=(au^2+2buv+cv^2)(a\alpha^2+2ba\beta+c\beta^2)$$

gives

$$X^2-\frac{au^2+2buv+cv^2}{a\alpha^2+2ba\beta+c\beta^2}\,Y^2=-\Delta,$$

or

$$X^2-AY^2=-\Delta,$$

when we write X for $au\alpha+b(u\beta+v\alpha)+cv\beta$, Y for $a\alpha^2+2ba\beta+c\beta^2$ Δ for $ac-b^2$, and assume

$$u\beta-v\alpha=\pm 1.$$

We may use, instead of the form (a, b, c), any equivalent form. Perhaps the most interesting case is when (a, b, c) belongs to the principal class. The equation may then be written

$$X^2-\frac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2}\,Y^2=-\Delta,$$

$$X=u\alpha+\Delta v\beta,\quad Y=\alpha^2+\Delta\beta^2,\quad u\beta-v\alpha=\pm 1,$$

and the resulting decomposition will be binomial.

In fact, the form $(q_n, -p_n, q_n A)$ is here identical with

$$[\alpha^2+\Delta\beta^2, \quad -(u\alpha+\Delta v\beta), \quad u^2+\Delta v^2],$$

and is transformed into $(1, 0, \Delta)$ by $\begin{pmatrix} \pm v, & u \\ \pm\beta, & \alpha \end{pmatrix}$.* It belongs, then, to the principal class. We shall, therefore, arrive at a representation of A of the form $P^2+\Delta Q^2$.

For simplicity's sake, and on account of the special interest of the case, I take Δ positive.

10. If $p^2+\Delta q^2$ be a factor of Y, so that

$$A = \frac{u^2+\Delta v^2}{(\alpha^2+\Delta\beta^2)(p^2+\Delta q^2)} = \frac{u^2+\Delta v^2}{(\alpha p+\Delta\beta q)^2+\Delta(\alpha q-\beta p)^2}$$

with the condition

$$u(\alpha q-\beta p)-v(\alpha p+\Delta\beta q) = \pm 1,$$

then $(p^2+\Delta q^2)^2 A = \dfrac{(u^2+\Delta v^2)(p^2+\Delta q^2)}{\alpha^2+\Delta\beta^2} = \dfrac{(up+\Delta vq)^2+\Delta(uq-vp)^2}{\alpha^2+\Delta\beta^2}$,

and the condition above written is

$$(up+\Delta vq)\beta-(uq-vp)\alpha = \mp 1,$$

so that $(p^2+\Delta q^2)^2 A$ is such a number that a similar decomposition of it will be obtained by developing its square root as a continued fraction.

We may therefore multiply A by the square of any factor of Y having the form $p^2+\Delta q^2$, Y being any integer value giving

$$X^2-AY^2 = -\Delta.$$

11. We have a condition in order that $\dfrac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2}$ may be an integer.

Let $u = \epsilon+\alpha z$, $v = \eta+\beta z$, where ϵ, η may or may not be $< \alpha, \beta$ respectively.

Then $$A = z^2+\frac{2z(\epsilon\alpha+\Delta\eta\beta)+\epsilon^2+\Delta\eta^2}{\alpha^2+\Delta\beta^2},$$

and we must solve in integers

$$2z(\epsilon\alpha+\Delta\eta\beta)-\zeta(\alpha^2+\Delta\beta^2) = -(\epsilon^2+\Delta\eta^2);$$

but $$(\epsilon\alpha+\Delta\eta\beta)^2-(\epsilon^2+\Delta\eta^2)(\alpha^2+\Delta\beta^2) = -\Delta,$$

and consequently

$$2\Delta z = (\epsilon\alpha+\Delta\eta\beta)(\epsilon^2+\Delta\eta^2)+m(\alpha^2+\Delta\beta^2),$$
$$\Delta\zeta = (\epsilon^2+\Delta\eta^2)^2+m(\epsilon\alpha+\Delta\eta\beta).$$

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* From Problem III., Article 2, of the Additions to Euler's Algebra, it appears that if $Ap^2-2Bpq+Cq^2$ is a minimum, $\frac{p}{q}$ is convergent to $\frac{B}{A}$. Hence $\frac{v}{\beta}$ is a convergent to $\frac{p_n}{q_n}$ in this case.

In order that these may be divisible by Δ, we must have further

$$m\alpha+\epsilon^3 \equiv 0 \pmod{\Delta}.$$

The condition $u\beta-v\alpha = \pm 1$ implies that $\alpha^2+\Delta\beta^2$ is odd, for if A is integer and $\alpha^2+\Delta\beta^2$ is even, $u^2+\Delta v^2$ must also be even, and $u\beta-v\alpha = \pm 1$ cannot hold.

If Δ is odd, m is even; if Δ is even and ϵ is odd, m is odd; if Δ is even and ϵ is even, m is even.

We now have the means of forming as many numbers as we please to illustrate the foregoing results. It will be as well, however, to give the values of P and Q belonging to the representation which will be given by the continued fraction.

The equation $$P^2+\Delta Q^2 = \frac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2}$$

gives $$(P+i\sqrt{\Delta}.\,Q)\,(\alpha-i\sqrt{A}.\,\beta) = u+i\sqrt{\Delta}.\,v,$$

and thence $$P\alpha+\Delta\beta Q = u, \quad Q\alpha-P\beta = v,$$

or $$P = \frac{\alpha u-\Delta\beta v}{\alpha^2+\Delta\beta^2}, \qquad Q = \frac{\beta u+\alpha v}{\alpha^2+\Delta\beta^2}$$
$$= \mp(uv-A\alpha\beta), \quad = \mp(v^2-A\beta^2).$$

If we substitute $\epsilon+\alpha z$ for u, and $\eta+\beta z$ for v, and make the expression homogeneous by means of $(\epsilon\beta-\eta\alpha)^2 = 1$, we get

$$2\Delta P = m\,(\alpha^2-\Delta\beta^2)+3\epsilon\eta\Delta\,(\alpha\eta-\epsilon\beta)+\epsilon^3\alpha-\Delta^2\eta^3\beta,$$
$$\Delta Q = m\alpha\beta+\epsilon^3\beta+\Delta\eta^3\alpha.$$

As an example, we may take one of the simplest cases. Let $\epsilon = 0$, $\eta = \alpha = 1$, $\beta = 2\beta_1$, $m = 2p\Delta$; then

$$A = \{p\,(1-4\Delta\beta_1^2)-\Delta\beta_1\}^2+\Delta\,\{4p\beta_1+1\}^2.$$

If $p = 3$, $\beta_1 = 2$, $\Delta = 3$, we have

$$A = 147^2+3\,.\,25^2 = 23484.$$

Developing $\sqrt{A}$, we have accordingly

Partial Quotient.	J.	Denominator.
153,	153,	75,
4,	147,	25,
12,	153,	3,
&c.	&c.	

Again, let $\epsilon = 5$, $\alpha = 6$, $\eta = \beta = 1$, $\Delta = 5$, $m = 10p$;

then $$A = (41p+105)^2+70p+180.$$

The development is

Partial Quotient.	$J.$	Denominator.
$41p+105,$	$41p+105,$	$70p+180,$
$1,$	$29p+75,$	$12p+31,$
$5,$	$31p+80,$	$60p+155,$
$1,$	$29p+75,$	$14p+36,$
$5,$	$41p+105,$	$5,$
	&c. &c.,	

and $$A = (31p+80)^2+5(12p+31)^2.$$

13. Let us suppose that Δ is the middle denominator belonging to $\sqrt{\Delta . A}$, we then have $\Delta X^2 - \frac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2} Y^2 = -1,$

with the conditions $u\alpha - \Delta v\beta = \pm 1,\ X = u\beta + v\alpha,\ Y = \alpha^2+\Delta\beta^2.$

Proceeding as before, we have, in order that $A = \frac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2}$ may be an integer,

$$2z = (\eta\alpha+\epsilon\beta)(\epsilon^2+\Delta\eta^2)+m(\alpha^2+\Delta\beta^2),$$
$$\zeta = (\epsilon^2+\Delta\eta^2)^2+m\Delta(\eta\alpha+\epsilon\beta),$$
$$A = \Delta z^2+\zeta,\quad u = \epsilon+\Delta\beta z,\quad v = \eta+\alpha z.$$

If Δ is odd, m must be even. For, $\alpha^2+\Delta\beta^2$ being odd, α or β is odd and the other even. If α is odd and β even, ϵ is odd, and η odd or even. If η is odd, $\epsilon^2+\Delta\eta^2$ is even; and if η is even, $\eta\alpha+\epsilon\beta$ is even. The reasoning is similar if α is even and β odd.

If Δ is even, α will be odd and ϵ odd, and m will be even or odd according as $\eta\alpha+\beta\epsilon$ is even or odd, or according as η, β are or are not of the same species.

From $\Delta A = P^2+\Delta Q^2$, we have

$$(P+i\sqrt{\Delta}Q)(\alpha-i\sqrt{\Delta}\beta) = \Delta v+i\sqrt{\Delta}u,$$
$$P = \Delta\frac{v\alpha-u\beta}{\alpha^2+\Delta\beta^2}, \qquad Q = \frac{u\alpha+\Delta v\beta}{\alpha^2+\Delta\beta^2},$$
$$= \pm\Delta(uv-A\alpha\beta), \qquad = \pm(u^2-\Delta A\beta^2);$$

and, writing $\epsilon+\Delta\beta z$ for u, $\eta+\alpha z$ for v, and making the expression homogeneous by $(\epsilon\alpha-\Delta\eta\beta)^2 = 1$, we get

$$2P = \Delta\{m(\alpha^2-\Delta\beta^2)+3\epsilon\eta(\epsilon\alpha-\Delta\eta\beta)-\epsilon^3\beta+\Delta\eta^3\alpha\},$$
$$Q = \Delta m\alpha\beta+\epsilon^3\alpha+\Delta^2\eta^3\beta.$$

14. We have still to consider the case when the middle denominator is even and equation (b) holds.

Let $\Delta = 2\Pi$, and the equation to be treated is

$$\Pi(\alpha u+2\beta v)^2 - \frac{2v^2+\Pi u^2}{\alpha^2+2\Pi\beta^2}(\alpha^2+2\Pi\beta^2)^2 = -2,$$

or $$\Pi X^2 - \frac{2v^2+\Pi u^2}{\alpha^2+2\Pi\beta^2}\, Y^2 = -2,$$

with $$\alpha v - \Pi\beta u = \pm 1.$$

In the same manner as before, we get

$$4z = (2\epsilon\beta+\eta\alpha)\,(2\epsilon^2+\Pi\eta^2)+m\,(\alpha^2+2\Pi\beta^2),$$
$$2\zeta = (2\epsilon^2+\Pi\eta^2)^2+m\Pi\,(2\epsilon\beta+\eta\alpha),$$
$$A = \Pi z^2+\zeta, \quad \epsilon\alpha-\Pi\beta\eta = \pm 1.$$

And if $\Pi A = P^2+2\Pi Q^2$ is the decomposition in question, we have also, in the same manner as before,

$$4P = \Pi\,\{m\,(\alpha^2-2\Pi\beta^2)+6\epsilon\eta\,(\alpha\epsilon-\Pi\eta\beta)+\Pi\eta^3\alpha-4\epsilon^3\beta\}$$
$$2Q = \Pi m\alpha\beta+2\epsilon^3\alpha+\Pi^2\eta^3\beta,$$

subject to the minor conditions necessary that A, P, and Q may be integers.

As examples of the last two cases, (1) let $\alpha=4$, $\epsilon=\eta=\beta=1$, $\Delta=5$, $m=2n$, so that

$$z = 21n+15, \quad \zeta = 50n+36, \quad 5A = (105n+75)^2+250n+180.$$

The development of $\sqrt{5A}$ is

Partial Quotient.	J.	Denominator.
$105n+76$,	$105n+76$,	$40n+29$,
5,	$95n+69$,	$50n+36$,
4,	$105n+75$,	5,
$42n+30$,	$105n+75$,	$50n+36$,
&c.	&c.,	

$$5A = (55n+40)^2+5\,(40n+29)^2,$$
$$95n+69-(40n+29) = 55n+40;$$

(2) let $\Pi=3$, $\epsilon=2$, $\alpha=\beta=\eta=1$, $m=3$, giving

$$z = 19, \quad \zeta = 83, \quad A = 19^2+83, \quad 3A = 3498.$$

The development of $\sqrt{3A}$ is

Partial Quotient.	J.	Denominator.
59,	59,	17,
6,	43,	97,
1,	54,	6,
18,	54,	97,
&c.	&c.,	

and $$3A = 42^2+6\,.\,17^2.$$

15. Since, if

$$A = \frac{u^2+\Delta v^2}{(\alpha^2+\Delta\beta^2)(p^2+\Delta q^2)} = \frac{u^2+\Delta v^2}{(\alpha p-\Delta\beta q)^2+\Delta\,(\alpha q+\beta p)^2}$$

complies with the condition

$$u(\alpha p - \Delta\beta q) - \Delta v(\alpha q + \beta p) = \pm 1,$$

then $$A' = \frac{(u^2+\Delta v^2)(p^2+\Delta q^2)}{\alpha^2+\Delta\beta^2} = \frac{(up-\Delta vq)^2+\Delta(uq+vp)^2}{\alpha^2+\Delta\beta^2} \cdot$$

complies with the condition

$$(up-\Delta qv)\,\alpha - \Delta(uq+vp)\,\beta = \pm 1;$$

it follows that, if $p^2+\Delta q^2$ is a factor of Y in some solution of $\Delta X^2 - AY^2 = \pm 1$, we may multiply A by an even power of that factor, and a similar decomposition can be effected of the augmented number. In the case of § 13, it appears, by § 8, that any odd factor of A may be constituted a factor of Y.

Hence, when Δ is the middle term, we may augment the index of any odd factor of A of the form $p^2+\Delta q^2$ by an even number. For example, $\sqrt{39}$ has 3 for the middle denominator, and satisfies $3x^2 - 13y^2 = -1$. The continued fraction gives the decomposition $6^2+3.1^2$. The factor 13 is 3.2^2+1 of the same form. Hence the continued fraction for $\sqrt{3.13^3}$ or $\sqrt{6591}$ gives a similar decomposition, namely, $54^2+3.35^2$.

16. Like considerations apply in the case of § 14; for if

$$A = \frac{2v^2+\Pi u^2}{(\alpha^2+2\Pi\beta^2)(p^2+2\Pi q^2)} = \frac{2v^2+\Pi u^2}{(\alpha p - 2\Pi q\beta)^2+2\Pi(\alpha q+\beta p)^2},$$

with $$v(\alpha p - 2\Pi q\beta) - \Pi(\alpha q+\beta p)\,u = \pm 1;$$

then $$A' = \frac{(2v^2+\Pi u^2)(p^2+2\Pi q^2)}{\alpha^2+2\Pi\beta^2} = \frac{\Pi(up+2vq)^2+2(vp-\Pi uq)^2}{\alpha^2+2\Pi\beta^2},$$

with $$(vp-\Pi uq)\,\alpha - \Pi(up+2vq)\,\beta = \pm 1.$$

Hence, if we augment the index of any odd factor of A having the form $p^2+2\Pi q^2$ by an even number, the number so changed, A', multiplied by Π, can be decomposed as before by taking the continued fraction for $\sqrt{\Pi A'}$.

17. When, as in the preceding articles, we take Δ positive, the decomposition into the form $P^2+\Delta Q^2$ can be made in a finite number of ways, only one of which is given by the process indicated. Similar considerations, however, to those by which other decompositions of the form P^2+Q^2 were arrived at, are applicable to the more general form ("Proceedings," Vol. ix., p. 187 *et seq.*). Having already occupied as much space as the subject probably deserves, I abstain from details which in great part would be a repetition of those given for the more limited case. It will be observed that the equation (§ 9) from which we set out by no means requires that the coefficient of Y^2 should be an integer in the first instance.

Following out this remark, we can see that another similar decomposition will be obtained by dividing the square root of A by a factor of the same form and $< \sqrt{A}$.

An example will be sufficient to explain the process.

Suppose we take $A = 22116 = 2^2 . 3 . 19 . 97$;

then $\frac{\sqrt{A}}{1}$ gives $21^2+3 . 85^2$, $\frac{\sqrt{A}}{4}$ gives $117^2+3 . 53^2$, $\frac{\sqrt{A}}{19}$ gives $93^2+3 . 67^2$, $\frac{\sqrt{A}}{97}$ gives $147^2+3 . 13^2$.

If we take factors which are multiples of 3, the same decompositions are obtained. So also, if we take $4 . 19$, the conjugate factor of 97 in $\frac{A}{3}$, we get again the decomposition given by $\frac{\sqrt{A}}{97}$.

18. We might now take n even, so that $(-)^n D = \Delta$, with much the same result; the decomposition would then be of the form $P^2 - \Delta Q^2$. The number of such decompositions is infinite when one is possible. I omit the discussion of this case, and confine myself to a very brief account of the forms corresponding to the second part of the second theorem of § 2.

The form (a, b, c) may be supposed to be of the principal class of improperly primitive forms. We then have

$$X^2 - \frac{2u^2+2uv+\frac{\Delta+1}{2}v^2}{2a^2+2a\beta+\frac{\Delta+1}{2}\beta^2} . Y^2 = -\Delta,$$

with $$u\beta - va = \pm 1.$$

And, writing $\frac{U}{V}$ for the coefficient of Y^2, we find that the form $(V, -X, U)$ is of the principal class of improperly primitive forms, Δ being of the form $4m-1$.

To make the coefficient of Y^2 integral, put $u = \epsilon + az$, $v = \eta + \beta z$; then

$$2\Delta z = \left(2\epsilon a + \epsilon\beta + \eta a + \frac{\Delta+1}{2}\eta\beta\right)\left(2\epsilon^2 + 2\epsilon\eta + \frac{\Delta+1}{2}\eta^2\right) + m\left(2a^2 + 2a\beta + \frac{\Delta+1}{2}\beta^2\right),$$

$$\Delta\zeta = \left(2\epsilon^2 + 2\epsilon\eta + \frac{\Delta+1}{2}\eta^2\right)^2 + m\left(2\epsilon a + \epsilon\beta + \eta a + \frac{\Delta+1}{2}\eta\beta\right),$$

$$A = z^2 + \zeta.$$

The condition follows

$$(2\epsilon+\eta)^3 + 2m(2a+\beta) \equiv 0 \pmod{4\Delta}.$$

If Δ is the middle denominator belonging to $\surd A$, the equation is

$$\Delta X^2 - \frac{2u^2+2uv+\frac{\Delta+1}{2}v^2}{2a^2-2a\beta+\frac{\Delta+1}{2}\beta^2} Y^2 = -1,$$

$$X = u\beta+va, \quad Y = 2a^2-2a\beta+\frac{\Delta+1}{2}\beta^2, \quad 2ua+va-u\beta-\frac{\Delta+1}{2}v\beta = \pm 1,$$

the form

$$\left\{\left(2a^2-2a\beta+\frac{\Delta+1}{2}\beta^2, \quad -\Delta(u\beta+va), \quad \Delta\left(2u^2+2uv+\frac{\Delta+1}{2}v^2\right)\right\},$$

reduces to $\left(2,\ 1,\ \frac{\Delta+1}{2}\right)$ by the substitution

$$\begin{pmatrix} 2u+v, & u+\frac{\Delta+1}{2}v \\ \beta, & a \end{pmatrix} = \pm 1,$$

and $$2z = (\epsilon\beta+\eta a)\left(2\epsilon^2+2\epsilon\eta+\frac{\Delta+1}{2}\eta^2\right)+m\left(2a^2-2a\beta+\frac{\Delta+1}{2}\beta^2\right),$$

$$\zeta = \left(2\epsilon^2+2\epsilon\eta+\frac{\Delta+1}{2}\eta^2\right)^2+m\Delta(\epsilon\beta+\eta a),$$

$$A = \Delta z^2+\zeta, \quad 2\epsilon a-\epsilon\beta+\eta a-\frac{\Delta+1}{2}\eta\beta = \pm 1.$$

Then, if $\eta = 0$, $a = \beta = \epsilon = 1$, $\Delta = 7$, $z = 2m+1$, $\zeta = 7m+4$, we have

$$7A = 7^2(2m+1)^2+7^2m+28;$$

and the development of $\surd 7A$ gives

Partial Quotient.	J.	Denominator.
$14m+8$,	$14m+8$,	$21m+13$,
1,	$7m+5$,	$7m+4$,
3,	$14m+7$,	7,
$4m+2$,	$14m+7$,	$7m+4$,

and $$7A = (7m+5)^2+(7m+5)(7m+4)+2(7m+4)^2.$$

It will be observed that, if we take $m = -1$, the decomposition is still affected, but the middle denominator is 4, not 7. In fact, in this case the coefficient of Y^2 is < 7, and the middle denominator is changed to 4, the smaller factor. This was to be expected by § 5, but otherwise we may take m negative. When the middle term is even, we can obtain corresponding expressions.

January 9th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Dr. J. Hopkinson, F.R.S., was admitted into the Society.

Letters from the Auditor, Prof. W. G. Adams, F.R.S., and from the Chief Accountant of the Bank of England, were read; and as the Treasurer's accounts were found to be correct, it was agreed that his Report should be adopted.

Prof. Cayley read a paper "On a Theorem in Elliptic Functions." Prof. Smith spoke upon a correction in Sohncke's Tables, and also upon a Modular Equation. Prof. Greenhill communicated a paper "On Coefficients of Induction and Capacity of two Electrified Spheres." Mr. Tucker read an abstract of a paper, by Prof. Lloyd Tanner, "On certain systems of Partial Differential Equations of the First Order with several dependent variables."

The following presents were received:—

"Memoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série," par M. G. Darboux (from "Journal de Mathématiques pures et appliquées," Tome iv., 1878, pp. 1–56; Deuxième Partie, Tome iv., 1878, pp. 377–416).

"Educational Times," January, 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome ii., Août, Sept., 1878; Paris.

"Bulletin des Sciences Mathématiques et Astronomiques: Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré," par M. G. Darboux.

"Proceedings of the Royal Society," Vol. xxviii., No. 190.

"Annali di Matematica," Serie 11ª, Tome ix., Fasc. 2, Nov. 1878; Milano.

"American Journal of Mathematics Pure and Applied," No. iv., Vol. i.; Baltimore.

"Atti della R. Accademia dei Lincei anno cclxxvi.," 1878–9, Serie terza. "Transunti," Vol. iii., Fasc i., Dic. 1878; Vol. iii., Fasc. 2°, Gennajo 1879; Roma, 1879.

Crelle's "Journal," 86 Band, drittes Heft; Berlin, 1878.

"Monatsbericht," Sept., Oct., 1879; Berlin, 1879.

"Tidsskrift for Mathematik," Fjerde Række, Anden Aargang, Første, Andet, Tredie, Fjerde, Femte, Sjette Hefte, udgifet af H. B. Zeuthen.

"Ueber die Erniedrigung der Modular-Gleichungen," von F. Klein, in München; and

"Ueber die Transformation siebenter Ordnung der elliptischen Funktionen," von demselben ("Math. Annalen," Bd. xiv.)

"Beiblätter zu den Annalen der Physik und Chemie," edited by G. and E. Wiedemann, Band iii., Stück i.; Leipzig, 1879.

"Analisi Matematica—sulle equazioni modulari," nota del S. C. F. Klein (read before R. Istituto Lombardo, 2 Gennajo, 1879).

A Theorem in Elliptic Functions. By Prof. Cayley.

[*Read January 8th*, 1879.]

The theorem is as follows: If $u+v+r+s=0$, then

$$\begin{aligned} &-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s \\ &+ \quad \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \\ &-\frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s \\ =&-\frac{k'^2}{k^2}. \end{aligned}$$

It is easy to see that, if a linear relation exists between the three products, then it must be this relation: for the relation must be satisfied on writing therein $v=-u$, $s=-r$, and the only linear relation connecting $\operatorname{sn}^2 u \operatorname{sn}^2 r$, $\operatorname{cn}^2 u \operatorname{cn}^2 r$, $\operatorname{dn}^2 u \operatorname{dn}^2 r$ is the relation in question

$$\begin{aligned} &-k'^2 \operatorname{sn}^2 u \operatorname{sn}^2 r \\ &+ \quad \operatorname{cn}^2 u \operatorname{cn}^2 r \\ &-\frac{1}{k^2} \operatorname{dn}^2 u \operatorname{dn}^2 r \\ =&-\frac{k'^2}{k^2}. \end{aligned}$$

A demonstration of the theorem was recently communicated to me by Mr. Glaisher, and this led me to the somewhat more general theorem

$$\begin{aligned} &-k'^2 \operatorname{sn}(\alpha+\beta) \operatorname{sn}(\alpha-\beta) \operatorname{sn}(\gamma+\delta) \operatorname{sn}(\gamma-\delta) \\ &+ \quad \operatorname{cn}(\alpha+\beta) \operatorname{cn}(\alpha-\beta) \operatorname{cn}(\gamma+\delta) \operatorname{cn}(\gamma-\delta) \\ &-\frac{1}{k^2} \operatorname{dn}(\alpha+\beta) \operatorname{dn}(\alpha-\beta) \operatorname{dn}(\gamma+\delta) \operatorname{dn}(\gamma-\delta) \\ =&-\frac{k'^2}{k^2} - \frac{2k'^2(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)(\operatorname{sn}^2\beta-\operatorname{sn}^2\delta)}{1-k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta \,.\, 1-k^2\operatorname{sn}^2\gamma\operatorname{sn}^2\delta}. \end{aligned}$$

In fact, writing herein $\alpha+\gamma=0$, that is $\gamma=-\alpha$, the right-hand side becomes $=0$; and the arcs on the left-hand side are $\alpha+\beta$, $\alpha-\beta$,

$-\alpha+\delta$, $-\alpha-\delta$, which represent any four arcs the sum of which is $= 0$.

Writing in the last mentioned equation x, y, z, w for the sn's of α, β, γ, δ respectively, also

$$\begin{aligned} P &= x^2-y^2, & P_1 &= z^2-w^2, \\ Q &= 1-x^2-y^2+k^2x^2y^2, & Q_1 &= 1-z^2-w^2+k^2z^2w^2, \\ R &= 1-k^2x^2-k^2y^2+k^2x^2y^2, & R_1 &= 1-k^2z^2-k^2w^2+k^2z^2w^2, \\ D &= 1-k^2x^2y^2, & D_1 &= 1-k^2z^2w^2, \end{aligned}$$

the equation is

$$-k'^2\frac{PP_1}{DD_1}+\frac{QQ_1}{DD_1}-\frac{1}{k^2}\frac{RR_1}{DD_1}=-\frac{k'^2}{k^2}-\frac{2k'^2(x^2-z^2)(y^2-w^2)}{DD_1},$$

that is,

$$-k'^2PP_1+QQ_1-\frac{1}{k^2}RR_1+\frac{k'^2}{k^2}DD_1+2k'^2(x^2-z^2)(y^2-w^2)=0;$$

and it is easy to verify that the terms of the orders 0, 1, 2, 3 and 4 in x^2, y^2, z^2, w^2 separately destroy each other; for instance, for the terms of the order 2, we have

$$\begin{aligned} &-k'^2(x^2-y^2)(z^2-w^2)+\{(x^2+y^2)(z^2+w^2)+k^2(x^2y^2+z^2w^2)\} \\ &\quad-\frac{1}{k^2}\{k^4(x^2+y^2)(z^2+w^2)+k_2(x^2y^2+z^2w^2)\} \\ &\quad+\frac{k'^2}{k^2}\{-k^2(x^2y^2+z^2w^2)\}+2k'^2(x^2-z^2)(y^2-w^2)=0, \end{aligned}$$

that is, $$\begin{aligned} &-k'^2(x^2-y^2)(z^2-w^2)+(1-k^2)(x^2+y^2)(z^2+w^2) \\ &\quad+(k^2-1-k'^2)(x^2y^2+z^2w^2)+2k'^2(x^2-z^2)(y^2-w^2)=0; \end{aligned}$$

or, omitting the factor k'^2, this is

$$\begin{aligned} &-(x^2-y^2)(z^2-w^2)+(x^2+y^2)(z^2+w^2) \\ &\quad-2(x^2y^2+z^2w^2)+2(x^2-z^2)(y^2-w^2)=0, \end{aligned}$$

as it should be.

The theorem in its original form was obtained by me as follows: using the elliptic coordinates p, q, r such that

$$\begin{aligned} \frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}&=1, \\ \frac{x^2}{a+q}+\frac{y^2}{b+q}+\frac{z^2}{c+q}&=1, \\ \frac{x^2}{a+r}+\frac{y^2}{b+r}+\frac{z^2}{c+r}&=1; \end{aligned}$$

or, what is the same thing,

$$-\beta\gamma x^2 = a+p \,.\, a+q \,.\, a+r,$$
$$-\gamma\alpha y^2 = b+p \,.\, b+q \,.\, b+r,$$
$$-\alpha\beta z^2 = c+p \,.\, c+q \,.\, c+r,$$

where α, β, γ denote $b-c$, $c-a$, $a-b$ respectively; then, treating r as a constant, the coordinates x, y, z will belong to a point on the ellipsoid

$$\frac{x^2}{a+r}+\frac{y^2}{b+r}+\frac{z^2}{c+r}=1,$$

and the differential equation of the right lines upon this surface is

$$\frac{dp}{\sqrt{a+p \,.\, b+p \,.\, c+p}} = \frac{dq}{\sqrt{a+q \,.\, b+q \,.\, c+q}}.$$

Take x_0, y_0, z_0 the coordinates of a point on the surface, and p_0, q_0 the corresponding values of p, q, so that

$$-\beta\gamma x_0^2 = a+p_0 \,.\, a+q_0 \,.\, a+r,$$
$$-\gamma\alpha y_0^2 = b+p_0 \,.\, b+q_0 \,.\, b+r,$$
$$-\alpha\beta z_0^2 = c+p_0 \,.\, c+q_0 \,.\, c+r,$$

then the equation of the tangent plane at the point (x_0, y_0, z_0) is

$$\frac{xx_0}{a+r}+\frac{yy_0}{b+r}+\frac{zz_0}{c+r}=1,$$

or, substituting for x^2, x_0^2, &c., their values, we have

$$-\frac{\beta\gamma xx_0}{a+r} = \sqrt{a+p \,.\, a+q \,.\, a+p_0 \,.\, a+q_0},\ \text{&c.},$$

and consequently the equation of the tangent plane is

$$\alpha\sqrt{a+p \,.\, a+q \,.\, a+p_0 \,.\, a+q_0}+\beta\sqrt{b+p \,.\, b+q \,.\, b+p_0 \,.\, b+q_0}$$
$$+\gamma\sqrt{c+p \,.\, c+q \,.\, c+p_0 \,.\, c+q_0} = -\alpha\beta\gamma,$$

the equation of a plane intersecting the ellipsoid in a pair of lines; hence this equation (containing in appearance the two arbitrary constants p_0 and q_0) is the integral of the proposed differential equation.

Writing

$$\mathrm{sn}^2 u = A\,(a+p),\quad \mathrm{cn}^2 u = B\,(b+p),\quad \mathrm{dn}^2 u = C\,(c+p),$$

the values of A, B, C, k are determined; and, assuming for q, p_0, q_0 the like forms with the arguments v, u_0, v_0, the differential equation becomes $du = dv$, having the integral $u-u_0 = v-v_0$; while the foregoing

integral equation, on reducing the constant coefficients contained therein, takes the form

$$\begin{aligned} &-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} u_0 \operatorname{sn} v_0 \\ &+ \quad \operatorname{cn} u \operatorname{cn} v \operatorname{cn} u_0 \operatorname{cn} v_0 \\ &-\frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} u_0 \operatorname{dn} v_0 \\ =&-\frac{k'^2}{k^2}, \end{aligned}$$

viz., this equation holds good if $u-u_0 = v-v_0$. And by a change of signs we have the theorem.

If, as above, $u+v+r+s=0$, the theorem gives a linear relation between the three products $\operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s$, $\operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s$, $\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s$, and regarding at pleasure the sn's, the cn's, or the dn's as rational, one of these products will be rational while the other two will be each of them a quadric radical; and hence, rationalising, we obtain an equation which contains the product in question linearly, and contains besides only the squares of the sn's, cn's, or dn's; that is, we have three such equations containing the three products respectively. Bringing to one side the terms which contain the product, and again squaring, we obtain an equation involving only the squares of the sn's, cn's, or dn's; but the three equations thus obtained represent, it is clear, one and the same rational equation, which may be expressed as an equation between the squares of the sn's, or of the cn's, or of the dn's at pleasure. This equation may be obtained, as I will show, from the ordinary addition-equations of the elliptic functions, but it is not obvious how to obtain from them the three equations involving the products respectively, and these last have the advantage of being of a degree which is the half of the equation which involves only the squared functions.

Write x, y, z, w for $\operatorname{sn} u$, $\operatorname{sn} v$, $\operatorname{sn} r$, $\operatorname{sn} s$ respectively; then, writing

$$\begin{aligned} A &= x\sqrt{1-y^2 \cdot 1-k^2y^2}, & \alpha &= z\sqrt{1-w^2 \cdot 1-k^2w^2}, \\ A' &= y\sqrt{1-x^2 \cdot 1-k^2x^2}, & \alpha' &= w\sqrt{1-z^2 \cdot 1-k^2z^2}, \\ P &= x^2-y^2, & \varpi &= z^2-w^2, \\ D &= 1-k^2x^2y^2, & \delta &= 1-k^2z^2w^2, \end{aligned}$$

we have $$\operatorname{sn}(u+v) = -\operatorname{sn}(r+s),$$

that is, $$\frac{A+A'}{D} = \frac{P}{A-A'} = -\frac{\alpha+\alpha'}{\delta} = -\frac{\varpi}{\alpha-\alpha'},$$

and consequently $$\begin{aligned} D\varpi &= -(\alpha-\alpha')(A+A'), \\ P\delta &= -(\alpha+\alpha')(A-A'); \end{aligned}$$

whence $$D\varpi - P\delta = 2(A\alpha' - A'\alpha),$$

that is, $$(z^2-w^2)(1-k^2x^2y^2)-(x^2-y^2)(1-k^2z^2w^2)$$
$$= 2\{xw\sqrt{1-y^2 \cdot 1-k^2y^2 \cdot 1-z^2 \cdot 1-k^2z^2}$$
$$-yz\sqrt{1-x^2 \cdot 1-k^2x^2 \cdot 1-w^2 \cdot 1-k^2w^2}\};$$

rationalising, we obtain, as mentioned above, an equation containing only the squares x^2, y^2, z^2, w^2, and which is thus of a degree twice that of the equation containing the product $xyzw$. I worked out in this way the equation in (x^2, y^2, z^2, w^2), but the calculation was lost, and the easier way of obtaining it is obviously by means of the equation involving $xyzw$.

We have, by the theorem,

$$-k'^2 xyzw$$
$$+\sqrt{1-x^2 \cdot 1-y^2 \cdot 1-z^2 \cdot 1-w^2}$$
$$-\frac{1}{k^2}\sqrt{1-k^2x^2 \cdot 1-k^2y^2 \cdot 1-k^2z^2 \cdot 1-k^2w^2} = -\frac{k'^2}{k^2},$$

that is,

$$k'^2(1-k^2xyzw) = k^2\sqrt{1-x^2 \cdot 1-y^2 \cdot 1-z^2 \cdot 1-w^2}$$
$$-\sqrt{1-k^2x^2 \cdot 1-k^2y^2 \cdot 1-k^2z^2 \cdot 1-k^2w^2};$$

and then, writing

$$P = x^2+y^2+z^2+w^2,$$
$$Q = x^2y^2+x^2z^2+x^2w^2+y^2z^2+y^2w^2+z^2w^2,$$
$$R = x^2y^2z^2+x^2y^2w^2+x^2z^2w^2+y^2z^2w^2,$$
$$S = x^2y^2z^2w^2,$$

and using $\sqrt{S}$ to denote the rational function $xyzw$, we have

$$k'^4(1-2k^2\sqrt{S}+k^4S)$$
$$= k^4(1-P+Q-R+S)$$
$$+1-k^2P+k^4Q-k^6R+k^8S$$
$$-2k^2\sqrt{(1-P+Q-R+S)(1-k^2P+k^4Q-k^6R+k^8S)};$$

or, if for a moment the radical is called $\sqrt{\Delta}$, then the factor k^2 divides out, and the equation becomes

$$2\sqrt{\Delta} = 2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S+2k'^4\sqrt{S},$$

whence

$$4(1-P+Q-R+S)(1-k^2P+k^4Q-k^6R+k^8S)$$
$$-\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\}^2$$
$$-4k'^8S$$
$$= -2k^4\sqrt{S}\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\}.$$

The factor k'^4 divides out, and omitting it we have

$$4Q-P^2-4(1+k^2)R+16k^2S+2k^2PR-4(k^2+k^4)PS-k^4R^2+4k^4QS$$
$$=-2\sqrt{S}\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\},$$

or, as this may also be written,

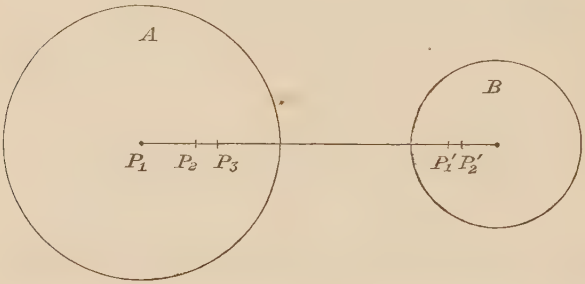
$$\{(-P^2+4Q-4R)+k^2(-4R+2PR+16S-4PS)$$
$$+k^4(-R^2+4QS-PS)\}$$
$$=-2\sqrt{S}\{2-P+k^2(-P+2Q-R)+k^4(-R+2S)\},$$

which is the required rational equation involving the product of the sn's.

Coefficients of Induction and Capacity of Two Electrified Spheres.

By A. G. Greenhill, M.A.

[*Read January 9th*, 1879.]



Let a, b be the radii of the spheres A and B, c the distance between the centres; and let A be charged to potential 1 and insulated, and let B be uninsulated, that is, at potential zero.

Then, with Thomson's notation (Papers on "Electrostatics and Magnetism," p. 87),

$$f_1=0,\quad p_1=a;\qquad f_n'=\frac{b^2}{c-f_n},\quad p_n'=-p_n\frac{f_n'}{b};$$

$$f_{n+1}=\frac{a}{c-f_n'},\quad p_{n+1}=-p_n'\frac{f_{n+1}}{a}.$$

Let
$$p_n=\frac{1}{P_n},\quad p_n'=-\frac{1}{P_n'};$$

therefore
$$f_n'=b\frac{P_n}{P_n'},\quad f_{n+1}=a\frac{P_n'}{P_{n+1}};$$

and therefore $$b\frac{P_n}{P'_n} = \frac{b^2}{c - a\dfrac{P'_{n-1}}{P_n}},$$

or $$cP_n = bP'_n + aP'_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

So also $$a\frac{P'_n}{P_{n+1}} = \frac{a^2}{c - b\dfrac{P_n}{P'_n}},$$

or $$cP'_n = aP_{n+1} + bP_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

From (1) and (2),

$$P_{n+1} - \frac{c^2 - a^2 - b^2}{ab} P_n + P_{n-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$P'_{n+1} - \frac{c^2 - a^2 - b^2}{ab} P'_n + P'_{n-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Therefore $$P_n = Ax^n + Bx^{-n},$$
$$P'_n = A'x^n + B'x^{-n},$$

where x is a root of the reciprocal quadratic equation

$$x^2 - \frac{c^2 - a^2 - b^2}{ab} x + 1 = 0.$$

If we put $x = e^{\varpi}$, then

$$e^{\varpi} + e^{-\varpi} = \frac{c^2 - a^2 - b^2}{ab},$$

or $$\cosh \varpi = \frac{c^2 - a^2 - b^2}{2ab}.$$

If we put $$\frac{b^2 + c^2 - a^2}{2bc} = \cosh \alpha$$

$$\frac{c^2 + a^2 - b^2}{2ca} = \cosh \beta,$$

then $$\sinh \alpha = \frac{h}{b}, \quad \sinh \beta = \frac{h}{a}, \quad \sinh \varpi = \frac{hc}{ab};$$

where $$h^2 = \frac{a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2}{4c^2},$$

and $$\alpha + \beta = \varpi.$$

Determining the constants A, B, A', B', we shall find

$$P_n = \frac{1}{h} \sinh (n\varpi - \alpha), \quad P'_n = \frac{1}{h} \sinh n\varpi,$$

and therefore $p_n = h \operatorname{cosech} (n\varpi - \alpha)$, $p'_n = -h \operatorname{cosech} n\varpi$.

Now q_{aa}, the coefficient of capacity of the sphere A, is the charge of the sphere A when charged to potential 1, the sphere B being at zero potential; and q_{bb}, the coefficient of induction of the spheres, is the charge of the sphere B.

Therefore $$q_{aa} = \Sigma p_n = h \sum_{n=1}^{n=\infty} \operatorname{cosech} (n\varpi - \alpha),$$

$$q_{ab} = \Sigma p'_n = -h \sum_{n=1}^{n=\infty} \operatorname{cosech} n\varpi.$$

If the sphere B had been insulated and charged to potential 1, and the sphere A had been uninsulated, we should have obtained the same value of q_{ab}, and we should find

$$q_{bb} = h \sum_{n=1}^{n=\infty} \operatorname{cosech} (n\varpi - \beta).$$

These are equivalent to the expressions given in Maxwell's *Electricity*, p. 216.

Put $\alpha - \beta = x$; also $\alpha + \beta = \varpi$, and put $q = e^{-\varpi}$, then

$$q_{aa} = h \sum_{n=1}^{n=\infty} \operatorname{cosech} \tfrac{1}{2} \{(2n-1)\,\varpi - x\},$$

$$q_{bb} = h \sum_{n=1}^{n=\infty} \operatorname{cosech} \tfrac{1}{2} \{(2n-1)\,\varpi + x\};$$

and, by a formula of elliptic functions,

$$q_{aa} - q_{bb} = hk \frac{K'}{\varpi} \operatorname{tanam} \left(K' \frac{x}{\varpi},\ k' \right),$$

where $$q = e^{-\varpi} = e^{-\pi \frac{K'}{K}}, \quad \text{or} \quad \varpi = \pi \frac{K'}{K}.$$

This is the formula for any two spheres not in contact, analogous to the formula for two spheres in contact,

$$E_a - E_b = \frac{\pi ab}{a+b} \tan \frac{\pi}{2} \frac{a-b}{a+b},$$

given on page 220 of Maxwell's *Electricity*.

When the spheres are in contact, $c = a+b$, $h = 0$, $\alpha = 0$, $\beta = 0$, $\varpi = 0$; but

$$lt \frac{h}{\varpi} = \frac{ab}{c} = \frac{ab}{a+b}; \quad lt \frac{\alpha}{\varpi} = \frac{a}{a+b}, \quad lt \frac{\beta}{\varpi} = \frac{b}{a+b};$$

and $$q_{aa} = \frac{ab}{a+b} \sum_{n=1}^{n=\infty} \frac{1}{n - \frac{a}{a+b}} = \frac{ab}{a+b} \sum_{n=0}^{n=\infty} \frac{1}{n + \frac{b}{a+b}},$$

$$q_{ab} = -\frac{ab}{a+b} \sum_{n=1}^{n=\infty} \frac{1}{n},$$

$$q_{bb} = \frac{ab}{a+b} \sum_{n=1}^{n=\infty} \frac{1}{n - \frac{b}{a+b}} = \frac{ab}{a+b} \sum_{n=0}^{n=\infty} \frac{1}{n + \frac{a}{a+b}}.$$

The series for q_{aa}, q_{ab}, q_{bb} are now infinite, but E_a and E_b are finite, for, the potential of each sphere being unity,

$$E_a = q_{aa} + q_{ab} = \frac{ab}{a+b} \sum_{n=1}^{n=\infty} \left(\frac{1}{n - \frac{a}{a+b}} - \frac{1}{n} \right),$$

$$E_b = q_{ab} + q_{bb} = \frac{ab}{a+b} \sum_{n=0}^{n=\infty} \left(\frac{1}{n - \frac{b}{a+b}} - \frac{1}{n} \right).$$

Now $$\Gamma(x) = \lim_{\mu=\infty} \mu^x \frac{1.2.3 \ldots \mu}{x(x+1)(x+2) \ldots (x+\mu)};$$

and therefore, denoting $\frac{d}{dx} \log \Gamma(x)$ by $\psi(x)$,

$$\psi(x) = \log \mu - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \ldots - \frac{1}{x+\mu}, \quad \mu = \infty.$$

Therefore $$E_a = \frac{ab}{a+b} \left\{ \psi(1) - \psi\left(\frac{b}{a+b}\right) \right\},$$

$$E_b = \frac{ab}{a+b} \left\{ \psi(1) - \psi\left(\frac{a}{a+b}\right) \right\}.$$

Since $$\Gamma\left(\frac{1-x}{2}\right) \Gamma\left(\frac{1+x}{2}\right) = \frac{\pi}{\cos \frac{1}{2}\pi x},$$

therefore $$\psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{1-x}{2}\right) = \pi \tan \tfrac{1}{2}\pi x,$$

and, putting $$x = \frac{a-b}{a+b},$$

$$\psi\left(\frac{a}{a+b}\right) - \psi\left(\frac{b}{a+b}\right) = \pi \tan\left(\frac{\pi}{2} \frac{a-b}{a+b}\right),$$

leading to the result

$$E_a - E_b = \frac{\pi ab}{a+b} \tan\left(\frac{\pi}{2} \frac{a-b}{a+b}\right).$$

[$\psi(1)$ is Euler's constant; these expressions for E_a and E_b in terms of the function ψ are given in Mr. Niven's paper read before this Society Dec. 1876, Vol. VIII., p. 78.]

Now $$\frac{d}{dx} \log \coth \tfrac{1}{2}(n\varpi - x) = \operatorname{cosech}(n\varpi - x),$$

and this suggests that we should put

$$\phi(x, q) = \prod_{n=1}^{n=\infty} \coth \tfrac{1}{2}(n\varpi - x)$$

$$= \prod_{n=1}^{n=\infty} \frac{1+e^{-n\varpi+x}}{1-e^{-n\varpi+x}}$$

$$= \prod_{n=1}^{n=\infty} \frac{1+q^n e^x}{1-q^n e^x}, \text{ where } q = e^{-\varpi},$$

and that we should also put

$$\psi(x, q) = \frac{d}{dx} \log \phi(x, q)$$

$$= \sum_{n=1}^{n=\infty} \operatorname{cosech}(n\varpi - x),$$

so that $$q_{aa} = h\psi(\alpha, q) = h\psi\left(\frac{\varpi+x}{2}, q\right),$$

$$q_{bb} = h\psi(\beta, q) = h\psi\left(\frac{\varpi-x}{2}, q\right).$$

The expression $\phi(x, q)$ is one of Heine's generalized hyper-geometric series, discussed on p. 97 of the second edition of Heine's *Kugelfunctionen*.

Analogous to the formula

$$\Gamma\left(\frac{1+x}{2}\right)\Gamma\left(\frac{1-x}{2}\right) = \frac{\pi}{\cos\frac{\pi}{2}x},$$

we have $$\phi\left(\frac{\varpi+x}{2}, q\right) = \prod_{n=1}^{n=\infty} \coth \tfrac{1}{2}\left(n\varpi - \frac{\varpi+x}{2}\right)$$

$$= \prod_{n=0}^{n=\infty} \coth \tfrac{1}{4}\{(2n+1)\varpi - x\},$$

and $$\phi\left(\frac{\varpi-x}{2}, q\right) = \prod_{n=0}^{n=\infty} \coth \tfrac{1}{4}\{(2n+1)\varpi + x\};$$

and therefore $$\phi\left(\frac{\varpi+x}{2}, q\right)\phi\left(\frac{\varpi-x}{2}, q\right)$$

$$= \prod_{n=0}^{n=\infty} \coth \tfrac{1}{4}\{(2n+1)\varpi - x\} \coth \tfrac{1}{4}\{(2n+1)\varpi + x\}$$

$$= \prod_{n=-\infty}^{n=\infty} \prod_{m=-\infty}^{m=\infty} \frac{x+(2n+1)\varpi+(2m+1)2i\pi}{x+(2n+1)\varpi+4mi\pi}$$

$$= \prod_{n=-\infty}^{n=\infty} \prod_{m=-\infty}^{m=\infty} \frac{K_1'\frac{x}{\varpi}+(2n+1)K_1'+(2m+1)iK_1}{K_1'\frac{x}{\varpi}+(2n+1)K_1'+2miK_1},$$

where $$\frac{K_1'}{K_1} = \frac{\varpi}{2\pi} = \frac{K'}{2K}.$$

Therefore $$\phi\left(\frac{\varpi+x}{2}, q\right)\phi\left(\frac{\varpi-x}{2}, \right)$$

$$= \frac{\Theta_1\left(K_1'\frac{x}{\varpi}, k_1'\right)}{\mathrm{H}_1\left(K_1'\frac{x}{\varpi}, k_1'\right)} = \frac{1}{\sqrt{k_1}}\frac{\mathrm{dn}\,K_1'\frac{x}{\varpi}}{\mathrm{cn}\,K_1'\frac{x}{\varpi}}$$

$$= \frac{1}{\sqrt{k_1}\,\mathrm{sn}\left(K_1'\frac{\varpi-x}{\varpi}, k_1'\right)},$$

analogous to $$\Gamma\left(\frac{1+x}{2}\right)\Gamma\left(\frac{1-x}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}(1-x)}.$$

Therefore $$q_{aa} - q_{bb} = h\psi\left(\frac{\varpi+x}{2}, q\right) - h\psi\left(\frac{\varpi-x}{2}, q\right)$$

$$= \tfrac{1}{2}h\frac{d}{dx}\log\frac{\mathrm{dn}\left(K_1'\frac{x}{\varpi}, k_1'\right)}{\mathrm{cn}\left(K_1'\frac{x}{\varpi}, k_1'\right)}$$

$$= \tfrac{1}{2}h\frac{K_1'}{\varpi}\frac{k_1^2\,\mathrm{sn}\left(K_1'\frac{x}{\varpi}, k_1'\right)}{\mathrm{cn}\left(K_1'\frac{x}{\varpi}, k_1'\right)\mathrm{dn}\left(K_1'\frac{x}{\varpi}, k_1'\right)}$$

$$= \frac{1}{2i}h\frac{K_1'}{\varpi}\frac{k_1^2\,\mathrm{sn}\left(iK_1'\frac{x}{\varpi}, k_1\right)\mathrm{cn}\left(iK_1'\frac{x}{\varpi}, k_1\right)}{\mathrm{dn}\left(iK_1'\frac{x}{\varpi}, k_1\right)};$$

which, by Landen's transformation,

$$= \frac{1}{2i}h\frac{K_1'}{\varpi}\frac{k_1^2}{1+k_1'}\,\mathrm{sn}\left(iK'\frac{x}{\varpi}, k\right)$$

$$= h\frac{K'}{\varpi}\frac{1-k_1'}{1+k_1'}\,\mathrm{tanam}\left(K'\frac{x}{\varpi}, k'\right)$$

$$= hk\frac{K'}{\varpi}\,\mathrm{tanam}\left(K'\frac{x}{\varpi}, k'\right),$$

as before.

The expression for q_{ab} is Lambert's series, and is given at the end of the "Fundamenta Nova."

With Heine's notation for the generalized hypergeometric series,

$$\phi(\alpha, \beta, \gamma, q, \xi) = 1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})} g^{\xi} + \dots,$$

we shall have $$\phi(x, q) = \phi\left(i\frac{\pi}{\varpi}, 1, 1, q, x\right).$$

The electrification at any point Q of the sphere A when insulated and charged to potential 1, the sphere B being uninsulated, is that due to the successive images P_1, P_2, P_3 ....

The electrification due to the image P_n

$$= p_n \frac{a^2 - f_n^2}{4\pi a} \frac{1}{P_n Q^3}$$

and this expanded in a series of zonal harmonics, of which the axis is the line drawn from the centre of the sphere A to the centre of the sphere B,

$$= \frac{p_n}{4\pi a^2} \sum_{i=0}^{i=\infty} (2i+1) \left(\frac{f_n}{a}\right)^i Q_i$$

$$= \frac{h}{4\pi a^2} \operatorname{cosech}(n\varpi - \alpha) \sum_{i=0}^{i=\infty} (2i+1) \left\{\frac{\sinh(n\varpi - \varpi)}{\sinh(n\varpi - \alpha)}\right\}^i Q_i;$$

and therefore the electrification of the sphere A

$$= \frac{h}{4\pi a^2} \sum_{n=1}^{n=\infty} \operatorname{cosech}(n\varpi - \alpha) \sum_{i=0}^{i=\infty} (2i+1) \left\{\frac{\sinh(n\varpi - \varpi)}{\sinh(n\varpi - \alpha)}\right\}^i Q_i.$$

Similarly, the electrification at any point of the sphere B

$$= -\frac{h}{4\pi a^2} \sum_{n=1}^{n=\infty} \operatorname{cosech} n\varpi \sum_{i=0}^{i=\infty} (2i+1) \left\{\frac{\sinh(n\varpi - \alpha)}{\sinh n\varpi}\right\}^i Q_i,$$

the axis of the zonal harmonics being now the line drawn from the centre of the sphere B to the centre of the sphere A.

To obtain the expressions for q_{aa}, q_{ab}, and q_{bb} as definite integrals, as given by Poisson,

$$q_{aa} = h \sum_{n=1}^{n=\infty} \operatorname{cosech}(n\varpi - \alpha)$$

$$= 2h\left(\frac{e^{-\beta}}{1-e^{-2\beta}} + \frac{e^{-\varpi-\beta}}{1-e^{-2\varpi-2\beta}} + \dots\right)$$

$$= 2h(e^{-\beta} + e^{-3\beta} + e^{-5\beta} + \dots + e^{-\varpi-\beta} + e^{-3\varpi-3\beta} + e^{-5\varpi-5\beta} + \dots) + \dots$$

and, adding vertical columns,

$$q_{aa} = 2h\left(\frac{e^{-\beta}}{1-e^{-\varpi}}+\frac{e^{-3\beta}}{1-e^{-3\varpi}}+\ldots\right)$$

$$= 2h\sum_{n=0}^{n=\infty}\frac{e^{-(2n+1)\beta}}{1-e^{-(2n+1)\varpi}}$$

$$= h\sum_{n=0}^{n=\infty}e^{-(2n+1)\beta}\left\{1+\frac{1+e^{-(2n+1)\varpi}}{1-e^{-(2n+1)\varpi}}\right\}$$

$$= h\frac{e^{-\beta}}{1-e^{-2\beta}}+h\Sigma e^{-(2n+1)\beta}\coth\tfrac{1}{2}(2n+1)\varpi.$$

Now $$2\int_0^\infty \coth\pi x\sin mx\,dx = \coth\tfrac{1}{2}m,$$

therefore $$q_{aa} = \tfrac{1}{2}h\operatorname{cosech}\beta+2h\int_0^\infty\coth\pi x\sum_{n=0}^{n=\infty}e^{-(2n+1)\beta}\sin(2n+1)\varpi x\,dx$$

$$= \tfrac{1}{2}h\operatorname{cosech}\beta+2h\int_0^\infty\coth\pi x\,\frac{e^{-\beta}(1+e^{-2\beta})\sin\varpi x\,dx}{1-2e^{-2\beta}\cos2\varpi x+e^{-4\beta}}$$

$$= \tfrac{1}{2}h\operatorname{cosech}\beta+2h\cosh\beta\int_0^\infty\frac{\coth\pi x\sin\varpi x\,dx}{\cosh2\beta-\cos2\varpi x}.$$

Similarly $$q_{bb} = \tfrac{1}{2}h\operatorname{cosech}\alpha+2h\cosh\alpha\int_0^\infty\frac{\coth\pi x\sin\varpi x\,dx}{\cosh2\alpha-\cos2\varpi x},$$

and $$q_{ab} = \tfrac{1}{2}\operatorname{cosech}\varpi+2h\cosh\varpi\int_0^\infty\frac{\coth\pi x\sin\varpi x\,dx}{\cosh2\varpi-\cos2\varpi x};$$

equivalent to the expressions given by Poisson.

On Certain Systems of Partial Differential Equations of the First Order with several Dependent Variables. *By* H. W. Lloyd Tanner, M.A.

[*Read January 9th*, 1879.]

The simplest systems considered in this paper are composed of equations such as

$$\frac{dz_1}{dx_1}+\frac{dz_2}{dx_2}+\ldots+\frac{dz_n}{dx_n}=0.$$

In the general case there are n independent variables, and $m(n-m)+1$ dependent variables. For the sake of symmetry, however, we assume

a larger number of dependent variables, and between these relations are given sufficient to reduce them to the number stated above. The system comprises $\frac{\lfloor n}{\lfloor m-1 \lfloor n-m+1}$ equations, and each equation consists of $n-m+1$ terms. The general solution of such a system expresses each of the dependent variables as a Jacobian determinant of $n-m$ arbitrary functions of $x_1 \ldots x_n$.

From the systems just described others are derived with their general solutions.

Some results* previously obtained by the author are in part simplified and in part generalized.

I.

1. The simplest case ($m = 1$) has been discussed by the author in the "Messenger of Mathematics."† There is a single equation

$$\frac{dz_1}{dx_1} + \frac{dz_2}{dx_2} + \ldots + \frac{dz_n}{dx_n} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the general solution is

$$(-)^{i+1} z_i = \frac{d\,(y_1 \ldots\ldots\ldots\ldots y_{n-1})}{d\,(x_1 \ldots x_{i-1},\ x_{i+1} \ldots x_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $y_1, \ldots y_{n-1}$ are arbitrary functions of $x_1 \ldots x_n$. That this is a solution of (1) is proved by substituting in (1). The expression on the left of (1) becomes

$$\begin{vmatrix} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \ldots\ldots & \frac{d}{dx_n} \\ \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \ldots\ldots & \frac{dy_1}{dx_n} \\ \ldots & \ldots & & \ldots \\ \frac{dy_{n-1}}{dx_1}, & \ldots\ldots\ldots & & \frac{dy_{n-1}}{dx_n} \end{vmatrix}$$

which vanishes identically.

That the system (2) is the *general* solution of (1) is proved, by first showing that the suppositions

$$(-)^{i+1} z_i = \lambda \frac{d\,(y_1 \ldots\ldots\ldots\ldots y_{n-1})}{d\,(x_1 \ldots x_{i-1},\ x_{i+1} \ldots x_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

leave $z_1 \ldots z_n$ arbitrary. Introducing these values into (1), it is found

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* Proc. London Math. Soc., Vol. IX., pp. 47—55.

† Vol. VII., pp. 107 *et sqq.*

that, to satisfy this equation, λ must be a function of $y_1 \dots y_{n-1}$ only; so that, when (1) is satisfied, (3) reduces to (2).

2. Before passing to the general investigation, it will perhaps be advantageous to deal with the case in which $m = 2$; as the discussion, otherwise unchanged, will be free from the complicated notation necessary when m is left undetermined.

In this case there are n equations :—

$$\left.\begin{aligned} \frac{dz_{12}}{dx_2} + \frac{dz_{13}}{dx_3} + \dots + \frac{dz_{1n}}{dx_n} &= 0 \\ \frac{dz_{21}}{dx_1} + \frac{dz_{23}}{dx_3} + \dots + \frac{dz_{2n}}{dx_n} &= 0 \\ \dots \quad \dots \quad \dots \quad \dots \\ \frac{dz_{n1}}{dx_1} + \frac{dz_{n2}}{dx_2} + \dots + \frac{dz_{n,n-1}}{dx_{n-1}} &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (4).$$

The $n(n-1)$ dependent variables z_{ij} are subject to conditions of the form

$$\left.\begin{aligned} z_{12} &= -z_{21} \\ z_{12} \cdot z_{34} - z_{13} \cdot z_{24} + z_{14} \cdot z_{23} &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (5).$$

If we suppose

$$z_{ii} = 0 \quad \dots\dots\dots\dots\dots\dots (5'),$$

the second condition (5) includes the first. In virtue of (5), only $2n-3$ of the z's are mutually independent. For instance, from

$$z_{12}, z_{13}, \dots\dots z_{1n},$$
$$z_{23}, \dots\dots z_{2n},$$

we can, by the second equation of (5), determine any other z.

The general solution of (4) is

$$z_{ij} = -z_{ji} = (-)^{i+j+1} \cdot \frac{d(y_1 \dots\dots\dots\dots\dots\dots y_{n-2})}{d(x_1 \dots x_{i-1}, x_{i+1} \dots x_{j-1}, x_{j+1} \dots x_n)} \quad \dots (6),$$
$$(i < j),$$

where $y_1, \dots y_{n-2}$ are $n-2$ arbitrary functions of $x_1 \dots x_n$. The solution may also be expressed by saying that z_{ij} is the coefficient of

$$\frac{d(\eta_1, \eta_2)}{d(x_i, x_j)},$$

in the expansion of

$$\frac{d(\eta_1, \eta_2, y_1 \dots y_{n-2})}{d(x_1, x_2, \dots\dots x_n)}$$

3. That the equations (6) do form a solution of (4) may be proved,

as in Art. 1, by substitution. If we replace z, in (4), by their values, the expressions on the left of (4) become the determinants of the matrix

$$\left\| \begin{matrix} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \ldots\ldots & \frac{d}{dx_n} \\ \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \ldots\ldots & \frac{dy_1}{dx_n} \\ \ldots & \ldots & & \ldots \\ \frac{dy_{n-2}}{dx_1}, & \ldots\ldots\ldots & & \frac{dy_{n-2}}{dx_n} \end{matrix} \right\|,$$

each of which vanishes identically.

The values of z given by (6) also satisfy the algebraical relations (5). To prove this, replace $\frac{d}{dx_1}$, ... $\frac{d}{dx_n}$, in the matrix last written, by a row of quantitative symbols a_1, ... a_n. It is known that, if two (independent) determinants of the matrix thus formed vanish, then all the determinants of the matrix vanish. For example, if

$$\left.\begin{aligned} a_2 \cdot z_{12} + a_3 \cdot z_{13} + a_4 \cdot z_{14} + \ldots + a_n \cdot z_{1n} &= 0 \\ a_1 \cdot z_{21} + a_3 \cdot z_{23} + a_4 \cdot z_{24} + \ldots + a_n \cdot z_{2n} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(7),$$

then also $a_1 \cdot z_{31} + a_2 \cdot z_{32} + a_4 \cdot z_{34} + \ldots + a_n \cdot z_{3n} = 0.$

But if from the first and second we eliminate a_3, we get

$$a_1 \cdot z_{21} \cdot z_{13} - a_2 \cdot z_{12} \cdot z_{23} + a_4 (z_{24} \cdot z_{13} - z_{14} \cdot z_{23}) + \&c. = 0,$$

which is consistent with the third equation only if we have

$$z_{13} \cdot z_{24} - z_{14} \cdot z_{23} = z_{12} \cdot z_{34},$$

or

$$z_{12} \cdot z_{34} - z_{13} \cdot z_{24} + z_{14} \cdot z_{23} = 0,$$

and the equations formed by replacing 4 herein by 5, ... n. But these are the equations (5).

It is to be noticed—and the remark will be useful in the sequel—that (5) are the sufficient conditions to ensure that, when the two equations (7) are satisfied, all the equations of the system are satisfied; upon the assumption that these two equations are independent, in other words that z_{12} does not vanish.

4. In proving that (6) is the general solution of (4), it is not possible to follow the course indicated in Art. 1. Instead of this, we shall discuss the equations of the preceding articles from a new point of view. We shall consider for the present that z_{ij} are known functions

of $x_1 \dots x_n$, and shall regard the equations (6) as equations from which $y_1 \dots y_{n-2}$ are to be determined. We assume that the given values of z_{ij} satisfy the conditions (4), (5); for, if they did not, no values of $y_1 \dots y_{n-2}$ could satisfy the system (6).

If y represent any of the quantities $y_1 \dots y_{n-2}$ or any function of them, then, identically,

$$\frac{d(y, y_1, \dots y_{n-2})}{d(x_1 \dots\dots x_n)} = 0.$$

This is a system which may be written

$$\left.\begin{aligned}
\Delta_1 y &= z_{12}\frac{dy}{dx_2} + z_{13}\frac{dy}{dx_3} + \dots + z_{1n}\frac{dy}{dx_n} = 0 \\
\Delta_2 y &= z_{21}\frac{dy}{dx_1} + z_{23}\frac{dy}{dx_3} + \dots + z_{2n}\frac{dy}{dx_n} = 0 \\
\Delta_3 y &= z_{31}\frac{dy}{dx_1} + z_{32}\frac{dy}{dx_2} + \dots + z_{3n}\frac{dy}{dx_n} = 0 \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}\right\} \quad \dots\dots\dots\dots(8).$$

In virtue of the conditions (5), this system of n equations reduces to two only (Art. 3); and these two may be the first and second if only z_{12} does not vanish. But two equations such as (8) involving n independent variables cannot have more than $n-2$ common solutions; and, in order that they may have so many, certain conditions must be satisfied. These conditions—sufficient as well as necessary—may be written [using the Δ's as defined in (8)],

$$\frac{1}{z_{12}} \cdot \Delta_1 \frac{z_{23}}{z_{21}} = \frac{1}{z_{21}} \cdot \Delta_2 \frac{z_{13}}{z_{12}},$$

where the 3 is to be replaced by 4, ... n in succession. Observe now that the Δ's are lineal functions of $\frac{d}{dx_1} \dots \frac{d}{dx_n}$; so that the equation just written may be reduced to the form

$$z_{12} \cdot \Delta_1 z_{23} - z_{12} \cdot \Delta_2 z_{13} - (z_{23} \cdot \Delta_1 - z_{13} \cdot \Delta_2)\, z_{12} = 0;$$

whence, since $\qquad z_{23} \cdot \Delta_1 - z_{13} \cdot \Delta_2 = -\, z_{12} \cdot \Delta_3,$

and z_{12} does not vanish,

$$\Delta_1 \cdot z_{23} - \Delta_2 \cdot z_{13} + \Delta_3 \cdot z_{12} = 0 \quad \dots\dots\dots\dots\dots(9).$$

But this equation, and others similar to it, are identically true in virtue

of (4). For, written at length and rearranged, the left side of (9) is

$$\begin{aligned}
&-z_{21}\frac{d}{dx_1}z_{13}+z_{31}\frac{d}{dx_1}z_{12}\\
&+z_{12}\frac{d}{dx_2}z_{23}+z_{32}\frac{d}{dx_2}z_{12}\\
&+z_{13}\frac{d}{dx_3}z_{23}-z_{23}\frac{d}{dx_3}z_{13}\\
&+z_{14}\frac{d}{dx_4}z_{23}-z_{24}\frac{d}{dx_4}z_{13}+z_{34}\frac{d}{dx_4}z_{12}\\
&+\&c.
\end{aligned}$$

But, differentiating (5) with respect to x_p, we get

$$\begin{aligned}
z_{14}\frac{d}{dx_p}z_{23}-z_{24}\frac{d}{dx_p}z_{13}+z_{34}\frac{d}{dx_p}z_{12}&\\
=-z_{23}\frac{d}{dx_p}z_{14}+z_{13}\frac{d}{dx_p}z_{24}-z_{12}\frac{d}{dx_p}z_{34}&.
\end{aligned}$$

Making this substitution for $p = 4, \ldots n$ in succession in the expansion of the left side of (9) just written, and again rearranging, we find

$$\left.\begin{aligned}
&\Delta_1 z_{23}-\Delta_2 z_{13}+\Delta_3 z_{12}\\
=&-z_{12}\left\{\frac{dz_{31}}{dx_1}+\frac{dz_{32}}{dx_2}+\ldots+\frac{dz_{3p}}{dx_p}+\ldots\right\}\\
&+z_{13}\left\{\frac{dz_{21}}{dx_1}+\frac{dz_{23}}{dx_3}+\ldots+\frac{dz_{2p}}{dx_p}+\ldots\right\}\\
&-z_{23}\left\{\frac{dz_{12}}{dx_2}+\frac{dz_{13}}{dx_3}+\ldots+\frac{dz_{1p}}{dx_p}+\ldots\right\}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(10),$$

and therefore vanishes identically by reason of (4).

Thus it appears that, because of (5), the system (8) reduces to two equations only; and because of (4) these two equations have $n-2$ common solutions, which indeed are found by integrating the integrable system

$$\begin{aligned}
z_{23}dx_1-z_{13}dx_2+z_{12}dx_3&=0,\\
z_{24}dx_1-z_{14}dx_2+z_{12}dx_4&=0,\\
\&c.&
\end{aligned}$$

Any set of $n-2$ solutions hence obtained will satisfy the equations (8) or (6).

5. The results of the last article have in effect proved that (6) is the general solution of (4). For it has been shown that, whatever $z_{ij}, \ldots$ may be—subject only to the conditions (4), (5)—values of $y_1 \ldots y_{n-2}$ can be found to satisfy the equations (6). But this is the same thing

as saying that every solution of (4), (5) is included in the form (6) when $y_1 \dots y_{n-2}$ are arbitrary, so that (6) is in fact the general solution of (4), (5).

6. At the commencement of Art. 4, it was remarked that the process employed to prove our solution to be general in the case of $m = 1$ was not applicable to the case in which $m = 2$. Indeed, if $z_{ij} \dots$ be taken as defined by (4), (5), the quantities $\lambda z_{ij}, \dots$ are not arbitrary; they are subject to certain conditions. If we write $z_{ij} \dots$ for $\lambda z_{ij}, \dots$, these may be written

$$\left.\begin{aligned} z_{ij}\left\{\frac{dz_{kl}}{dx_1}+\frac{dz_{k2}}{dx_2}+\dots+\frac{dz_{kn}}{dx_n}\right\} \\ +z_{ki}\left\{\frac{dz_{jl}}{dx_1}+\frac{dz_{j2}}{dx_2}+\dots+\frac{dz_{jn}}{dx_n}\right\} \\ +z_{jk}\left\{\frac{dz_{il}}{dx_1}+\frac{dz_{i2}}{dx_2}+\dots+\frac{dz_{in}}{dx_n}\right\}=0 \\ z_{ij}=-z_{ji},\ z_{ii}=0 \\ z_{ij}z_{kl}+z_{ik}z_{lj}+z_{il}z_{jk}=0 \\ i, j, k, l = 1, 2, \dots n \end{aligned}\right\} \quad \dots\dots\dots\dots(11);$$

and, as is suggested by the manner of obtaining this system, the general solution is

$$z_{ij}=-z_{ji}=\lambda\frac{d\,(y_1, y_2, \dots\dots\dots\dots\dots\dots y_{n-2})}{d\,(x_1 \dots x_{i-1}, x_{i+1}, \dots x_{j-1}, x_{j+1}, \dots x_n)},\ (i<j) \dots(12),$$

where λ, $y_1, \dots y_{n-2}$ are arbitrary functions of $x_1, \dots x_n$.

The results already obtained enable us to prove this without much difficulty.

First, to prove that (12) is a solution of (11), it is only necessary to observe that the differential equation in (11) is, by (10), merely a transformation of

$$\frac{1}{z_{12}}\Delta_1\frac{z_{23}}{z_{21}}=\frac{1}{z_{21}}\Delta_2\frac{z_{13}}{z_{12}},$$

or

$$\left\{\frac{d}{dx_2}+\frac{z_{13}}{z_{12}}\cdot\frac{d}{dx_3}+\dots\right\}\frac{z_{23}}{z_{21}}=\left\{\frac{d}{dx_1}+\frac{z_{23}}{z_{21}}\cdot\frac{d}{dx_3}+\dots\right\}\frac{z_{13}}{z_{12}},$$

and therefore is not affected by a change of $z_{ij}, \dots$ into $\lambda z_{ij}, \dots$ for all values of i, j. The same is true of the other equations in (11), for they are homogeneous. But when the z's have the values given in (6), they satisfy the system (11). Hence also, when they have the values (12), namely when they are all multiplied by one and the same arbitrary function of $x_1 \dots x_n$, they still satisfy (11), though they no longer satisfy (4).

To prove the generality of the solution, it is only necessary to repeat

the arguments of Art. 4. For the determination of $y_1, \ldots y_{n-2}$, we have the same equations (8) as before. We prove that (8) reduces to two equations by the algebraical relations in (11). The differential equation (11) shows that these two equations have $n-2$ common solutions, $y_1, \ldots y_{n-2}$, as we see from the identity (10). Hence, as in Art. 5, we infer that (12) is the general solution of (11).

II.

7. In the general case, every z has m subscripts. Should two of these subscripts be identical, the z vanishes; and a transposition of two subscripts changes the sign of the z. It will be convenient in most cases to represent $m-1$ of the subscripts by a single symbol. Thus $1, 2, \ldots m$, irrespective of order, might be represented by $\alpha 1$, or $\beta 2 \ldots$, or μm; where, for instance, α represents the subscripts $23 \ldots m$; $\beta = 13 \ldots m$; $\mu = 12 \ldots m-1$. To avoid ambiguity, we suppose that in any such group the subscripts are in order of magnitude.

By the use of this kind of abbreviation, it is easy to write down the algebraical relations connecting the z's. Let us select any m subscripts, say $1, 2, \ldots m$, and suppose these expressed as above. Choose also a second group πp, where p is a single subscript, π is a group of $m-1$ subscripts. Then the relations we seek may be written

$$z_{\pi 1} \cdot z_{\alpha p} - z_{\pi 2} \cdot z_{\beta p} + \ldots \pm z_{\pi m} \cdot z_{\mu p} \mp z_{\pi p} \cdot z_{\mu m} = 0 \quad \ldots\ldots\ldots(13),$$

the upper or lower sign being used according as m is odd or even.

By reason of these conditions, the number of mutually independent z's is reduced to $m(n-m)+1$. For instance, we can find the whole series if we know the z's which have the subscripts given in the following scheme, omitting those in brackets,

$$\left.\begin{array}{l} \alpha 1 \;;\; \alpha, m+1 \;;\; \ldots\; \alpha, p \;;\; \ldots\; \alpha n \\ (\beta 2) \;;\; \beta, m+1 \;;\; \ldots\; \beta, p \;;\; \ldots\; \beta n \\ \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ (\mu m) \;;\; \mu, m+1 \;;\; \ldots\; \mu, p \;;\; \ldots\; \mu n \end{array}\right\} \quad \ldots\ldots\ldots\ldots(14).$$

The groups $(\beta 2) \ldots (\mu m)$ are merely new arrangements of $\alpha 1$, and are therefore known when $\alpha 1$ is given. The other groups form m rows, and each row contains $n-m$ members; so that (14) includes $m(n-m)+1$ groups. All of these are different. For in the first row (we leave $\alpha 1$ out of sight for the present) the subscript 1 does not occur, in the second row no group comprises 2, and so on; but these subscripts occur in every other row. At the same time, every member of a column contains a subscript, $m+1, \ldots p, \ldots n$, which does not appear in any other column. Thus, for two groups to be identical, they must belong to the same column and to the same row.

The attempt to form any relation (13) between the z's indicated by (14) will fail. For, suppose that p is one of the numbers 1, 2, ... m; say $p = 1$. Then $z_{\beta p} = z_{\beta 1} = 0, \ldots, \quad z_{\mu p} = z_{\mu 1} = 0,$

and (13) reduces to $z_{\pi 1}\, z_{\alpha 1} \pm z_{\pi 1}\, z_{\mu m} = 0$

or $z_{\alpha 1} \pm z_{\mu m} = 0.$

But the subscript μm is not included in the scheme (14).

If, on the other hand, p is not one of the numbers 1, 2, ... m, then π must be one of the groups $\alpha, \beta, \ldots \mu$; for, otherwise, πp would be outside the scheme. But if, for example, we put $p = \alpha$, we reproduce the equation $z_{\alpha 1} \pm z_{\mu m} = 0.$

Thus it appears that the conditions (13) leave $m\,(n-m)+1$ of the z's mutually independent. It remains to show that, given the z's denoted in (14), all the other z's can be found by the help of (13). Now the groups in (14) comprise all arrangements of $m-1$ of the numbers 1, 2, ... m, with one of the numbers greater than m. Hence, if in (13) we take π as composed of $m-2$ of the first m numbers and one number greater than m, the groups $\pi 1$, $\pi 2$, ... πm, are all included in (14), so that the first m terms of (13) are known. Also $z_{\mu m}$ is known; thus (13) becomes an equation to determine $z_{\pi p}$, a z which in its subscripts includes two of the numbers $m+1$, $m+2$, ... n. By properly choosing p, π, we can thus determine all such z's. But now we may take π as a group involving two of the numbers above m. For $z_{\pi 1}, z_{\pi 2}, \ldots z_{\pi m}$, whose subscripts involve only two numbers above m, are all known. In this case (13) becomes an equation for determining all the z's whose subscripts involves three numbers greater than m. By repetition of this process, it is evident that we can successively form all the z's if we know those given in (14). Hence only $m\,(n-m)+1$ are mutually independent.

8. The system of differential equations to be solved is

$$\frac{dz_{\pi 1}}{dx_1} + \frac{dz_{\pi 2}}{dx_2} + \ldots + \frac{dz_{\pi n}}{dx_n} = 0 \quad \ldots\ldots\ldots\ldots\ldots (15),$$

where π is any combination of 1, 2, ... n taken $m-1$ together. There are therefore $\dfrac{\lfloor n}{\lfloor m-1 \rfloor n-m+1}$ equations in the system. In each equation there are $n-m+1$ terms,—viz., one term for each subscript not included in π. There are $\dfrac{\lfloor n}{\lfloor m-1 \rfloor n-m}$ dependent variables. These being, however, subject to the conditions of Art. 7, $m\,(n-m)+1$, and no more of them, are mutually independent.

The general solution is

$$z_{ij\ldots k} = (-)^{i+j+\ldots k+\frac{m\,.\,m+1}{2}} \frac{d\,(y_1 \ldots y_{n-m})}{d\,(x_1 \ldots\ldots x_n)}, \quad i<j\ldots<k \ldots\ldots (16),$$

where, in the range $x_1 \dots x_n$, $x, x_j, \dots x_k$ are omitted, and $y_1, \dots y_{n-m}$ are arbitrary functions of $x_1 \dots x_n$. The sign is + or − according as $i, j \dots k$, $1, \dots n$, is reducible to the natural order $1, 2, \dots n$ by an even or an odd number of transpositions. The general solution may also be expressed by saying that $z_{ij\dots k}$ is the coefficient of $\frac{d(\eta_1, \dots \eta_m)}{d(x_i, x_j \dots x_k)}$ in the expansion of $\frac{d(\eta_1, \dots \eta_m, y_1, \dots y_{n-m})}{d(x_1, \dots \dots \dots x_n)}$.

Of course the conditions $i < j < \dots < l$ are not assumed here.*

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* *Note by Prof. Cayley.*

The general result may be illustrated as follows. Take, for instance, $n = 7$; the problem is to find the most general values of $\left(\frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = \right) 35$ quantities $y_{123}, \dots y_{567}$ (functions of the independent variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7$) satisfying certain algebraical relations, and also certain differential relations.

It is convenient to assume that any one, for instance y_{123}, of the 35 y's, may be written under the different forms

$$y_{123}, y_{231}, y_{312}, -y_{132}, -y_{321}, -y_{213},$$

(and so in other cases, the sign being always + or − according to the number of interchanges,) that is, y_{231}, $-y_{132}$, &c. each mean y_{123}, and the number of the y's is thus = 35, as stated above.

The algebraical relations are the whole system of algebraical relations implied in the assumption that the y's are severally equal to the determinants formed out of the matrix

$$\left\| \begin{matrix} \frac{dP}{dx_1}, & \frac{dP}{dx_2} & \dots\dots & \frac{dP}{dx_7} \\ \frac{dQ}{dx_1}, & \frac{dQ}{dx_2} & \dots\dots & \frac{dQ}{dx_7} \\ \frac{dR}{x_1}, & \frac{dR}{dx_2} & \dots\dots & \frac{dR}{dx_7} \\ \frac{dS}{dx_1}, & \frac{dS}{dx_2} & \dots\dots & \frac{dS}{dx_7} \end{matrix} \right\|$$

where P, Q, R, S are arbitrary functions of $x_1, x_2 \dots x_7$, viz., $y_{123} = \pm$ the determinant formed with columns 4, 5, 6, 7 out of this matrix, or in a convenient notation

$$y_{123} = \pm \frac{\delta(P, Q, R, S)}{\delta(x_4, x_5, x_6, x_7)}, = \pm \frac{\delta(P, Q, R, S)}{\delta_{4567}(x, x, x, x)},$$

the $\pm$ sign being that answering to the arrangement 123 . 4567, considering as derived from 1234567 assumed to be positive; for the equation written down, the sign is of course +.

The differential equations are the *whole* system of differential equations such as

$$\frac{d}{dx_3} y_{123} + \frac{d}{dx_4} y_{124} + \frac{d}{dx_5} y_{125} + \frac{d}{dx_6} y_{126} + \frac{d}{dx_7} y_{127} = 0,$$

and the theorem is that the *general* values of y, satisfying the algebraical relations, and also these differential equations, are the above values

$$y_{123} = \pm \frac{\delta(P, Q, R, S)}{\delta_{4567}(x, x, x, x)},$$

where P, Q, R, S are arbitrary functions of $x_1, x_2, \dots x_7$.

9. As in Art. 3, it can be proved that the forms of z given by (16) satisfy the given equations (15); these reducing in fact to the identities—

$$\left\| \begin{matrix} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \ldots\ldots & \frac{d}{dx_n} \\ \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \ldots\ldots & \frac{dy_1}{dx_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{dy_{n-m}}{dx_1}, & \ldots\ldots & \ldots\ldots & \frac{dy_{n-m}}{dx_n} \end{matrix} \right\| = 0.$$

That the conditions (13) are also fulfilled, may be proved in the same manner as in Art. 3. In the matrix just written we replace $\frac{d}{dx_1}, \ldots \frac{d}{dx_n}$ by a row of quantities, $A_1, \ldots A_n$. Suppose that

$$\frac{d\,(y_1, \ldots\ldots y_{n-m})}{d\,(x_{m+1}, \ldots\ldots x_n)}$$

does not vanish. Then it is known that, if all the determinants vanish which are formed by taking each of the first m columns in turn with the last $n-m$ columns, every determinant of the matrix vanishes. That is, if

$$\left.\begin{matrix} A_1 \cdot z_{\alpha 1} + A_{m+1} \cdot z_{\alpha, m+1} + \ldots + A_p \cdot z_{\alpha p} + \ldots + A_n \cdot z_{\alpha n} = 0 \\ A_2 \cdot z_{\beta 2} + A_{m+1} \cdot z_{\beta, m+1} + \ldots + A_p \cdot z_{\beta p} + \ldots + A_n \cdot z_{\beta n} = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ A_m \cdot z_{\mu m} + A_{m+1} \cdot z_{\mu, m+1} + \ldots + A_p \cdot z_{\mu p} + \ldots + A_n \cdot z_{\mu n} = 0 \end{matrix}\right\},$$

then also $A_1 \cdot z_{\pi 1} + A_2 \cdot z_{\pi 2} + \ldots + A_p \cdot z_{\pi p} + \ldots + A_n \cdot z_{\pi n} = 0.$

Here $\alpha, \beta, \ldots \mu, \pi$ have the meanings attributed to them in Art. 7. But this implies that, multiplying the first m equations by $\lambda_1, \lambda_2, \ldots \lambda_m$ respectively, and adding, we shall, upon suitable choice of $\lambda_1, \ldots \lambda_m$, produce the last equation. Suppose this done, and compare coefficients of $A_1, A_2, \ldots A_m$. Thus

$$\lambda_1 z_{\alpha 1} = z_{\pi 1}, \quad \lambda_2 z_{\beta 2} = z_{\pi 2}, \quad \ldots \lambda_m z_{\mu m} = z_{\pi m}.$$

Also, comparing the coefficients of A_p,

$$\lambda_1 z_{\alpha p} + \lambda_2 z_{\beta p} + \ldots + \lambda_m z_{\mu p} = z_{\pi p}.$$

Replace herein λ_1, λ_2, &c. by their values just found, remembering that $z_{\alpha 1} = -z_{\beta 2} = \ldots = \pm z_{\mu m}$, and we get

$$z_{\pi 1} \cdot z_{\alpha p} - z_{\pi 2} \cdot z_{\beta p} + \ldots \pm z_{\pi m} \cdot z_{\mu p} \mp z_{\pi p} \cdot z_{\mu m} = 0,$$

which is (13).

Thus the proof is complete that (16) is a solution of (15), (13).

10. We now regard (16) as a system of equations for the determination of $y_1, \ldots y_{n-m}$. It is assumed that the z's satisfy the conditions

(15), (13), because these have just been proved essential to the existence of such forms for z as are given by (16). The functions $y_1, \ldots y_{n-m}$ satisfy the equations, found as in Art. 4,

$$\left.\begin{aligned}
\Delta_\alpha y &= z_{\alpha 1}\frac{dy}{dx_1} + \ldots + z_{\alpha p}\frac{dy}{dx_p} + \ldots + z_{\alpha n}\frac{dy}{dx_n} = 0 \\
\Delta_\beta y &= z_{\beta 2}\frac{dy}{dx_2} + \ldots + z_{\beta p}\frac{dy}{dx_p} + \ldots + z_{\beta n}\frac{dy}{dx_n} = 0 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\Delta_\mu y &= z_{\mu m}\frac{dy}{dx_m} + \ldots + z_{\mu p}\frac{dy}{dx_p} + \ldots + z_{\mu n}\frac{dy}{dx_n} = 0 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}\right\} \ldots\ldots\ldots(17).$$

and

$$z_{\pi 1}\frac{dy}{dx_1} + \ldots + z_{\pi p}\frac{dy}{dx_p} + \ldots + z_{\pi n}\frac{dy}{dx_n} = 0$$

represents any equation of the system.

We first show that, supposing $z_{\alpha 1}$ does not vanish, all the equations of the system are algebraical consequences of the first m equations, viz. (17). This follows at once from the fact that the z's satisfy (13). For, multiplying the first equation by $z_{\pi 1}$, the second by $-z_{\pi 2}$, the third by $z_{\pi 3}$, and so on, and adding the results, we get, by reason of (13),

$$z_{\alpha 1}\left\{ z_{\pi 1}\frac{dy}{dx_1} + z_{\pi 2}\frac{dy}{dx_2} + \ldots + z_{\pi p}\frac{dy}{dx_p} + \ldots \right\} = 0.$$

But, by hypothesis, $z_{\alpha 1}$ does not vanish. Hence the expression within brackets vanishes. But this is equivalent to saying, any equation of the system is satisfied.

11. The system (17), consisting of m mutually independent equations, and involving n independent variables, cannot have more than $n-m$ different solutions, nor can it have so many unless Boole's conditions are fulfilled. One of these conditions is

$$\frac{1}{z_{\alpha 1}} \cdot \Delta_\alpha \frac{z_{\beta q}}{z_{\beta 2}} = \frac{1}{z_{\beta 2}} \cdot \Delta_\beta \frac{z_{\alpha q}}{z_{\alpha 1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

and the rest are similar. But this condition becomes

$$z_{\alpha 1} \cdot \Delta_\alpha \cdot z_{\beta q} - z_{\alpha 1} \cdot \Delta_\beta \cdot z_{\alpha q} - \{ z_{\beta q} \cdot \Delta_\alpha - z_{\alpha q} \cdot \Delta_\beta \}\, z_{\alpha 1} = 0 \quad \ldots\ldots(18').$$

In this expression let us select the terms involving $\frac{d}{dx_p}$. They are

$$z_{\alpha 1} \cdot z_{\alpha p} \cdot \frac{d}{dx_p} \cdot z_{\beta q} - z_{\alpha 1} \cdot z_{\beta p} \cdot \frac{d}{dx_p} \cdot z_{\alpha q} - (z_{\beta q} \cdot z_{\alpha p} - z_{\alpha q} \cdot z_{\beta p}) \cdot \frac{d}{dx_p} \cdot z_{\alpha 1}.$$

To simplify this expression we revert to (13), and assume therein

$\pi = 34 \ldots mq$. Then

$$z_{\pi 3} = z_{\pi 4} = \ldots = z_{\pi m} = 0,$$
$$z_{\pi 1} = z_{34\ldots mq1} = \pm z_{134\ldots mq} = \pm z_{\beta q},$$
$$z_{\pi 3} = z_{34\ldots mq2} = \pm z_{234\ldots mq} = \pm z_{\alpha q},$$

the upper or lower sign being taken in each case according as m is odd or even. Then (13) becomes

$$z_{\beta q} \cdot z_{\alpha p} - z_{\alpha q} \cdot z_{\beta p} \mp z_{\pi p} \cdot z_{\alpha 1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

the upper and lower signs still corresponding to m odd or even. Hence the expression we have to simplify becomes, on dividing by $z_{\alpha 1}$ which does not vanish,

$$z_{\alpha p} \cdot \frac{d}{dx_p} \cdot z_{\beta q} - z_{\beta p} \cdot \frac{d}{dx_p} \cdot z_{\alpha q} \mp z_{\pi p} \cdot \frac{d}{dx_p} \cdot z_{\alpha 1}.$$

But, differentiating (19) with respect to x_p, it appears that this last expression is identically equal to

$$-z_{\beta q} \cdot \frac{d}{dx_p} \cdot z_{\alpha p} + z_{\alpha q} \cdot \frac{d}{dx_p} \cdot z_{\beta p} \pm z_{\alpha 1} \cdot \frac{d}{dx_p} \cdot z_{\pi p}.$$

Summing this for all values of p from 1 to n inclusive, we get, for the value of the left side of (18′),

$$\left.\begin{aligned} &-z_{\beta q}\left\{\frac{dz_{\alpha 1}}{dx_1} + \ldots + \frac{dz_{\alpha p}}{dx_p} + \ldots + \frac{dz_{\alpha n}}{dx_n}\right\} \\ &+z_{\alpha q}\left\{\frac{dz_{\beta 2}}{dx_2} + \ldots + \frac{dz_{\beta p}}{dx_p} + \ldots + \frac{dz_{\beta n}}{dx_n}\right\} \\ &+z_{\mu m}\left\{\frac{dz_{\pi 1}}{dx_1} + \ldots + \frac{dz_{\pi p}}{dx_p} + \ldots + \frac{dz_{\pi n}}{dx_n}\right\} \end{aligned}\right\} \ldots\ldots\ldots\ldots (20),$$

which identically vanishes by (15), proving that the conditions (18) are satisfied. Hence the system (17) for y admits of $n-m$ integrals found from the equations

$$z_{\alpha 1}\, dx_p - z_{\alpha p}\, dx_1 + z_{\beta p}\, dx_2 - \ldots \pm z_{\mu p}\, dx_m = 0.$$

In fine, then, it appears that, whatever the z's may be, we can always express them in the form (16), provided only the conditions (13), (15) are satisfied; in other words, no restriction upon the forms of z is imposed by (16) which is not also implied in (13), (15). In Art. 9, on the other hand, it has been shown that the equations (16) do not include any forms of z excluded by (13), (15). Thus the two definitions—the one given by (13), (15), the other by (16)—are precisely equivalent. We may express this by saying that (16) is the general

solution of (13), (15); or, looking at the matter from another standpoint, that (13), (15) are the necessary and sufficient conditions that the transformations (16) should be possible.

12. No difficulty arises in extending the results of Art. 6 to the general case. Indeed, the reasoning of that article may be applied to the present case almost without change. Only it is desirable to modify the notation employed in (20) so as to put in evidence the relation between the symbols α, β, π, q. For this purpose we represent by ρ the group $34 \ldots m$, whence we get

$$\alpha = 2\rho, \ \beta = 1\rho, \ \pi = \rho q,$$

$$\beta q = 1\rho q, \ \alpha q = 2\rho q, \ \mu m = 12\rho = \rho 12.$$

Also observe that

$$z_{\beta q} \cdot \frac{dz_{\alpha p}}{dx_p} = z_{1\rho q} \cdot \frac{dz_{2\rho p}}{dx_p} = z_{\rho 1 q} \cdot \frac{dz_{\rho 2 p}}{dx_p},$$

since the same change of sign is effected in both factors. Thus the equation formed by expressing that (20) vanishes, takes the symmetrical form,

$$z_{\rho 2 q} \Sigma_p \frac{dz_{\rho 1 p}}{dx_p} - z_{\rho 1 q} \Sigma_p \frac{dz_{\rho 2 p}}{dx_p} + z_{\rho 12} \Sigma_p \frac{dz_{\rho q p}}{dx_p} = 0 \quad \ldots\ldots\ldots\ldots (21).$$

Supposing the z's to be still subject to (13), the general solution of the system of equations formed from (21) by making ρ, q take all values, is

$$z_{ij \ldots k} = (-)^{i+j+\ldots+k} \lambda \frac{d(y_1, \ldots y_{n-m})}{d(x_1, \ldots x_n)} \quad \ldots\ldots\ldots\ldots\ldots (22),$$

where, in the range $x_1, \ldots x_n$, $x_i, x_j \ldots x_k$ are omitted, and λ, $y_1, \ldots y_{n-m}$ are arbitrary functions of $x_1, x_2, \ldots x_n$.

III.

13. When m becomes greater than $n-m$, and especially when the latter is small, the notation for the z's employed in the previous articles may be conveniently replaced by a complementary notation. Instead of writing m subscripts, we should write the $n-m$ subscripts that would have been omitted. To avoid confusion we shall mark the z's with an accent when this new notation is employed.

It is necessary to make some convention as to the order of subscripts to a z' which is equivalent to a given z; and it obviously suggests itself that in

$$z_{ij \ldots k} = \pm z'_{\iota\kappa \ldots \lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

the upper or lower sign should be used according as the order

$i, j, \ldots k, \iota, \kappa, \ldots \lambda$ is reducible to $1, 2, \ldots n$ in an even or an odd number of transpositions.

The propriety of this convention is confirmed by its effect upon the transformation of (16). Suppose that $ij \ldots k \iota\kappa \ldots \lambda$ is reducible to the order $12 \ldots n$ by an even number of transpositions, so that in (23) the upper sign is used. Then, from the way in which the sign of (16) was determined, it follows that

$$z_{ij\ldots k} = + \frac{d(y_1 \ldots y_{n-m})}{d(x_\iota, x_\kappa, \ldots x_\lambda)}.$$

If, on the other hand, the reduction requires an odd number of transpositions, so that the lower sign is used in (23), we should have

$$z_{ij\ldots k} = - \frac{d(y_1 \ldots y_{n-m})}{d(x_\iota, x_\kappa, \ldots x_\lambda)}.$$

In each case
$$z'_{\iota\kappa\ldots\lambda} = \frac{d(y_1 \ldots y_{n-m})}{d(x_\iota, x_\kappa, \ldots x_\lambda)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

14. The transformation of equations such as (13) or (21), where the subscripts of the z's include group symbols, depends upon the following considerations. Let
$$z_\pi = z'_\rho,$$
so that
$$\pi\rho = 1, 2, \ldots n.$$

(The sign of equality is here used to imply that each of the two groups is reducible to the other in an even number of transpositions.) Also, α being another symbol representing a group of subscripts, let

$$z_{\pi\alpha} = z'_\sigma.$$

Then
$$\pi\alpha\sigma = 1, 2, \ldots n = \pi\rho,$$
$$\therefore \alpha\sigma = \rho.$$

This means that the subscript σ is formed from the subscript ρ by removing from the left of the latter the group α.

For instance, let $n = 6$, $\pi = 5$; then

$$z_\pi = z_5 = z'_{12346}.$$

Hence
$$z_{\pi 12} = z'_{346}.$$

Again, to transform $z_{\pi 36}$, observe that

$$z'_{12346} = z'_{36214},$$

so that
$$z_{536} = z_{\pi 36} = z'_{214}.$$

If any numbers are common to π, α,

$$z_{\pi\alpha} = 0.$$

Hence, if we have to remove from the subscript of z' a number not in it, that z' vanishes. Say, for example, we wish to transform $z_{\pi 5}$, π itself

containing 5. In the subscript of z' corresponding to z_π there is no 5 to be removed; so that we infer the z' vanishes.

15. Consider now the case in which $n-m=1$. In the equation (15) the group symbol π comprises $m-1$, that is $n-2$, numbers.

Suppose $$z_\pi = z'_{ij};$$

then $$z_{\pi i} = z'_j; \quad z_{\pi j} = -z'_i,$$

and all the other z''s vanish. Thus (15) becomes

$$\frac{dz'_j}{dx_i} - \frac{dz'_i}{dx_j} = 0,$$

and the system consists of all the equations that can be formed by giving i, j all values from 1 to n inclusive. From (24) the general solution of the system is (since $n-m=1$)

$$z'_i = \frac{dy_1}{dx_i},$$

where y_1 is an arbitrary function of $x_1 \ldots x_n$.

Take now the equation (21). The group symbol ρ comprises $m-2$, that is $n-3$, numbers. Let

$$z_\rho = z_{12q}.$$

Then $$z_{\rho 2q} = z'_1, \; z_{\rho 12} = z'_q, \; z_{\rho 13} = 0, \; \ldots \; z_{\rho 1q} = -z'_2 \ldots,$$

&c. &c.,

so that (21) becomes

$$z'_1\left(\frac{dz'_q}{dx_2} - \frac{dz'_2}{dx_q}\right) + z'_2\left(-\frac{dz'_q}{dx_1} + \frac{dz'_1}{dx_q}\right) + z'_q\left(\frac{dz'_2}{dx_1} - \frac{dz'_1}{dx_2}\right) = 0,$$

and the general solution of the whole system is, by (22),

$$z'_i = \lambda \frac{dy_1}{dx},$$

where λ, y_1 are arbitrary functions of $x_1 \ldots x_n$.

16. Next suppose $n-m=2$.

In this case it is necessary to take the equations (13) into account. Let us put $z_\pi = z'_{123}$, so that $z_{\pi 1} = z'_{23}$, $z_{\pi 2} = -z'_{13}$, $z_{\pi 3} = z'_{12}$, and all the others vanish. Then put $p=5$, and let

$$z_{\alpha 1} = -z_{\beta 2} = z_{\gamma 3} = \text{&c.} = z'_{45},$$

$$\therefore \; z_\alpha = z'_{145}, \text{ and } z_{\alpha 5} = z'_{14},$$

$$z_\beta = -z'_{215}, \quad \therefore \; z_{\beta 5} = -z'_{24},$$

$$z_\gamma = z'_{315}, \quad \therefore \; z_{\gamma 5} = z'_{34}.$$

Substituting these values in (13), it becomes

$$z'_{23} \cdot z'_{14} - z'_{13} \cdot z'_{24} + z'_{12} \cdot z'_{34} = 0.$$

The other conditions may be got in the same way, or may be inferred by changing the suffixes. It will be observed that the relations are the same as were obtained in the case of $m = 2$; and there is no difficulty in proving that the relations for z' may be obtained by the process described in Art. 7.

The differential equations (15) become

$$\frac{dz'_{jk}}{dx_i} - \frac{dz'_{ik}}{dx_j} + \frac{dz'_{ij}}{dx_k} = 0 \ (i, j, k = 1 \dots n),$$

and the general solution of this system is

$$z'_{ij} = \frac{d(y_1, y_2)}{d(x_i, x_j)}.$$

The system (21) is replaced by

$$z'_{il} \left\{ \frac{dz'_{kl}}{dx_i} - \frac{dz'_{jl}}{dx_k} + \frac{dz'_{jk}}{dx_l} \right\}$$
$$- z'_{jl} \left\{ \frac{dz'_{kl}}{dx_i} - \frac{dz'_{il}}{dx_k} + \frac{dz'_{ik}}{dx_l} \right\}$$
$$+ z'_{kl} \left\{ \frac{dz'_{jl}}{dx_i} - \frac{dz'_{il}}{dx_k} + \frac{dz'_{ij}}{dx_l} \right\} = 0,$$

$$(i, j, k, l = 1, \dots n,)$$

and the general solution of the system is

$$z'_{ij} = \lambda \frac{d(y_1, y_2)}{d(x_i, x_j)},$$

where λ, y_1, y_2, are arbitrary functions of $x_1 \dots x_n$.

IV.

17. Some further notice seems to be required of the fact that a process for proving the generality of the solution (16), which was successfully used in the case of $m = 1$, failed when $m = 2$. In the case of $m = 1$, we had

$$z_i = (-)^{i-1} \frac{d(y_1 \dots y_{n-1})}{d(x_1 \dots x_{i-1} x_{i+1} \dots x_n)},$$

so that z_i involves $n-1$ arbitrary functions of $x_1 \dots x_n$. If we introduce a new arbitrary function λ, as a coefficient to the Jacobian, the z's would involve n arbitrary functions of n variables $x_1 \dots x_n$, and, as we have seen, become themselves arbitrary.

In the case of $m = 2$ on the other hand, the z's involve only $n-2$ arbitrary functions. Hence, by the introduction of one new arbitrary function, they become dependent upon $n-1$ arbitrary functions, so that they cannot be themselves arbitrary functions of the n variables $x_1 \dots x_n$. But the question arises, whether we cannot, by introducing *two* (or, generally, m) new arbitrary functions into the values of z, make these arbitrary. If so, how are they to be distributed? The question is not without importance. For it will be remembered that the equations (10), (21), and the general solutions thereof, were suggested by the introduction of a supplementary arbitrary function such as had, in the case of $m = 1$, made z_i arbitrary. Judging from analogy, one would be led to suppose that, by introducing r new functions in a suitable manner, then we should get forms of z which for $m = r$ would be unconditioned, but which when $m > r$ would satisfy certain systems of differential equations; these systems would moreover be of the same form for all values of m greater than r [compare (21) for instance].

Nor are there wanting indications of the answers to be given to these two questions:—How are the supplementary functions to be distributed? and to what forms of differential equations are we thus led? In fact, in the case in which $n-m = 1$, a complete answer to each question has been given; and, as far as it goes, it confirms the remarks made above. We have already deduced, as particular cases of our general results, that the system

$$\frac{dz_i}{dx_j} - \frac{dz_j}{dx_i} = 0 \ (i, j = 1, 2 \dots n) \dots\dots\dots\dots\dots (25)$$

implies $$z_i = \frac{dy_1}{dx_i};$$

and that

$$z_i \left\{ \frac{dz_j}{dx_k} - \frac{dz_k}{dx_j} \right\} - z_j \left\{ \frac{dz_i}{dx_k} - \frac{dz_k}{dx_i} \right\} + z_k \left\{ \frac{dz_i}{dx_j} - \frac{dz_j}{dx_i} \right\} = 0 \dots (26)$$

$$(i, j, k = 1, 2 \dots n)$$

implies $$z_i = \lambda \frac{dy_1}{dx_i};$$

the two statements in each case being precisely equivalent. But, as is well known, these results can be extended. For instance, the system

$$\left\{ \frac{dz_i}{dx_j} - \frac{dz_j}{dx_i} \right\} \left\{ \frac{dz_k}{dx_l} - \frac{dz_l}{dx_k} \right\}$$
$$- \left\{ \frac{dz_i}{dx_k} - \frac{dz_k}{dx_i} \right\} \left\{ \frac{dz_j}{dx_l} - \frac{dz_l}{dx_j} \right\}$$
$$+ \left\{ \frac{dz_i}{dx_l} - \frac{dz_l}{dx_i} \right\} \left\{ \frac{dz_j}{dx_k} - \frac{dz_k}{dx_j} \right\} = 0 \dots\dots\dots\dots (27)$$

is precisely equivalent to this other system

$$z_i = \lambda \frac{dy_1}{dx_i} + \frac{dy_2}{dx_i}.$$

If now from (27) we form a new system in the same way as (26) was derived from (25), this will be equivalent to

$$z_i = \lambda_1 \frac{dy_1}{dx_i} + \lambda_2 \frac{dy_2}{dx_i}.$$

This process may be repeated until z_i involves n arbitrary functions $\lambda_1, y_1, \lambda_2, y_2, \ldots$, when it becomes itself arbitrary.

These results suggest what appears to be a very probable answer to the questions proposed. We infer that, when $m = 2$,

$$\pm z = \lambda \frac{d(y_1, y_{m+2} \ldots y_n)}{d(x_1 \ldots\ldots x_n)} + \frac{d(y_2, y_{m+2} \ldots y_n)}{d(x_1 \ldots\ldots x_n)}$$

is arbitrary, involving as it does n arbitrary functions $\lambda, y_1, y_2, y_4 \ldots y_n$. When, however, m is greater than 2, z ceases to be arbitrary, and satisfies a system of equations analogous to (27). If we represent the equation (15) by the symbol

$$(\pi) = 0,$$

one of the equations of the system would be

$$(\rho ij)(\rho kl) - (\rho ik)(\rho jl) + (\rho il)(\rho jk) = 0 \quad \ldots\ldots\ldots\ldots \quad (28),$$

where, of course, ρ is a symbol for a group containing $m-3$ letters.

Again, the supposition

$$z = \lambda_1 \frac{d(y_1, y_{m+2} \ldots y_n)}{d(x_1 \ldots\ldots x_n)} + \lambda_2 \frac{d(y_2, y_{m+2} \ldots y_n)}{d(x_1 \ldots\ldots x_n)}$$

leaves z arbitrary unless $m > 3$. If this inequality holds, then z would satisfy equations derived from (28), just as (26) is obtained from (25).

And so on.

I have ventured to make these remarks without a formal proof (a deficiency I hope soon to amend), because I cannot help thinking that the evidence in favour of them is of the strongest. In the first place, when z is a Jacobian, we have the equations (15) for the whole range of m. When z is proportional to a Jacobian, we have the equations (21) from $m = 2$ onwards; and the persistence of its form is very remarkable. Then, there is the *à priori* probability that, z being composed of $n-m$ arbitrary functions, the proper introduction of m more arbitrary functions should make z an arbitrary function of the n independent variables—a probability which becomes a certainty in the extreme cases $m = 1$, $m = n-1$. At one end of the series ($m = n-1$) we have the

complete set of systems of differential equations formed in succession by an alternating rule, and a set of corresponding general solutions which can also be formed in succession by an alternating process. The first step in each of these processes has been proved to hold good for all values of m. Moreover, it is difficult to believe that the persistence of form noted above should not hold good of the higher systems; and there is a confirmation of the opposite hypothesis in the fact that the differential equations formed by analogy break off at the right place. For example, in (28) we have a group symbol, ρ, involving $m-3$ numbers; so that m in this case cannot be less than 3, which is the limit otherwise arrived at. Add to these the close analogy between Jacobians and differential coefficients, and a body of evidence is completed which is scarcely less convincing than actual proof.

18. In a paper,—"On Partial Differential Equations of the first order with several Dependent Variables,"—published in the Society's "Proceedings,"* I have discussed the conditions that

$$\Sigma z_{ij\ldots k}\frac{d(u_1,\ldots u_m)}{d(x_i x_j \ldots x_k)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

should be expressible in the form

$$\lambda\frac{d(y_1,\ldots y_{n-m},\ u_1,\ldots u_m)}{d(x_1 \ldots\ldots\ldots\ldots x_n)}=0.$$

The conditions there obtained were left in a form equivalent to (18); and it seems desirable to draw attention to the simpler form (21), (13) in which they can be expressed. The conditions that the left side of (29) should itself be a Jacobian, *i.e.*, not merely a product of a Jacobian into an arbitrary function, are given by the systems (13), (15). The same sets of conditions are necessary and sufficient to ensure that a system of simultaneous equations such as (29) should reduce to the form

$$\lambda\frac{d(y_1\ldots y_r,\ u_1,\ldots u_m)}{d(x_1\ldots\ldots\ldots\ldots x_n)}=0,$$

where $m+r<n$.

Should the remarks of Art. 17 be confirmed, the theory of such equations as (29) will be considerably simplified: for these results would enable us to effect transformations of expressions such as that on the left of (29) in manner not unlike the transformations of

$$\Sigma y_1 dx_1,$$

first effected by Pfaff.

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* Vol. IX., pp. 41—61.

February 13th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Sir James Cockle, F.R.S., was admitted into the Society, and Mr. R. Hargreaves, M.A., Fellow of St. John's College, Cambridge, and Prof. W. E. Story, Ph.D. Leipsic, were proposed for election.

The following communications were made :—

"On the number of Conics which satisfy Five Independent Conditions," by M. Halphen (communicated by Dr. Hirst).

"Construction of Magic Squares," by Sir J. Cockle, F.R.S. (Messrs. Cayley, Henrici, Hart, Harley, and others took part in a discussion on the communication.)

"Notes on Frames," by Prof. Henrici, F.R.S.

"On a Modular Equation," and "On the Formula for four Abelian Functions answering to the Formula for four Theta Functions," by Prof. H. J. S. Smith, F.R.S.

"Quaternion Proof of Minding's Theorem," by Mr. J. J. Walker.

The following presents were made to the Society :—

"Proceedings of the Royal Society," Vol. xxviii., No. 192.

"Educational Times," February, 1879.

"Monatsbericht," Nov. 1878; Berlin, 1879.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2^{e} Série, Tome iii, 1er cahier; Paris, 1878.

"Theorie des Arithmetisch-geometrischen Mittels aus vier Elementen," von C. W. Borchardt (from the "Abhandlungen der K. Akad. der Wissenschaften zu Berlin," 1878); Berlin, 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^{e} Série, Tome ii., Nov. 1878; Paris, 1878.

"Bulletin de la Société Mathématique de France," Tome vii., No. 1; Paris 1879.

"A collection of Problems on Plane Geometrical Drawing, including Problems on a few of the Higher Plane Curves, &c.," by E. F. Mondy, A.R.S.M., Prof. of Drawing and Lecturer in Metallurgy in the Imperial College of Engineering, Tokei, Japan; Tokei, 1878: from the Author.

"Separataftryk af Archiv for Mathematik og Naturvidenskab," udgivet af S. Lie, W. Müller og G. O. Sars, containing :—

"Theorie der Transformations-Gruppen," Abhandlung i., ii., iii., iv.

"Bestimmung aller Gruppen einer zweifach ausgedehnten Punkt-Mannigfaltigkeit und Sätze über Minimal-Flächer," von Sophus Lie.

Sur le nombre des Coniques qui, dans un plan, satisfont à cinq Conditions projectives et indépendantes entre elles.

Par M. Halphen.

[*Read February* 13*th*, 1879.]

1. J'ai précédemment résolu le problème qui consiste à trouver le nombre des coniques faisant partie d'un système et satisfaisant, d'autre part, à une condition projective et indépendante de ce système. Le problème actuel est lié à ce dernier de la manière la plus naturelle; il s'y rattache aussi historiquement. Si cinq conditions séparées sont imposées à une conique, on peut réunir quatre d'entre elles pour définir un système, et ramener ainsi le second problème au premier. C'est ce qu'a fait M. Chasles; c'est ce que je fais ici.

2. M. Chasles a donné le nom de *conditions élémentaires* aux deux conditions les plus simples qu'on puisse imposer à une conique dans un plan, celle de passer par un point, celle de toucher une droite. Je désignerai ces deux conditions par les lettres P et D. Dans mon précédent mémoire, j'ai introduit la notion nécessaire d'une catégorie nouvelle de conditions élémentaires, dont je rappelle la définition :

Soient, sur une droite L, trois points y, z, t; soient x, x' les points où L est rencontrée par une conique a. Soit r la différence des deux rapports anharmoniques (y, z, t, x), (y, z, t, x') *dans un ordre déterminé, le même pour les deux cas.*

Soient aussi, par un point l, trois droites Y, Z, T; soient X, X' les tangentes menées par l à la conique a, et enfin R la différence des deux rapports anharmoniques (Y, Z, T, X), (Y, Z, T, X').

J'appelle condition élémentaire *imposée à la conique a, celle qui est définie par la relation*

$$r^{2q} = kR^{2p},$$

dans laquelle k, p, q sont des nombres donnés, p, q étant entiers, positifs et premiers entre eux.

Les deux nombres p, q étant seuls appelés à jouer un rôle dans les formules, je désignerai cette condition par le symbole (p, q). Mais on ne doit pas perdre de vue les éléments descriptifs L, l qui figurent dans la définition, et qui sont les arbitraires de la condition (p, q). On ne manquera pas de remarquer que la condition (p, q) a pour corrélative (q, p).

3. Dans le mémoire déjà cité, j'ai prouvé *qu'une condition quelconque est équivalente à la somme d'un nombre limité de conditions élémentaires.* Je rappelle la signification de cet énoncé. Soit

$$\Phi \equiv aP + bD + c(p, q) + c'(p', q') + \ldots$$

l'équivalent de la condition Φ, a, b, c, $c'\ldots$ étant des nombres entiers et positifs. Étant donné un système quelconque, soient, pour ce système, μ le nombre des coniques qui satisfont à la condition P, ν le nombre de celles qui satisfont à la condition D, ρ le nombre de celles qui satisfont à (p, q), ρ' à (p', q'), etc. Le nombre des coniques de ce même système qui satisfont à Φ est $a\mu + b\nu + c\rho + c'\rho' + \ldots$

Soient cinq conditions quelconques imposées à une conique. Prenons une condition élémentaire dans chacun de leurs cinq équivalents. Soit σ le nombre des coniques qui satisfont à ces cinq conditions élémentaires. Formons toutes les combinaisons analogues, et faisons la somme de tous les nombres analogues à σ. Cette somme est le nombre des coniques qui satisfont aux cinq conditions données. Ainsi le problème est ramené à celui-ci : *trouver le nombre des coniques qui satisfont à cinq conditions élémentaires.* En outre, l'hypothèse que les conditions, mises sous forme projective, sont indépendantes entre elles, entraîne pour ce problème particulier l'hypothèse précise suivante : *pour chacune des cinq conditions élémentaires, la droite L et le point l doivent être considérés comme arbitraires.*

4. Une formule unique et simple résout le problème, moyennant une convention que j'ai déjà introduite antérieurement, et que je rappelle. On étendra la notation (p, q) aux conditions P, D, en faisant

$$P \equiv (0, \tfrac{1}{2}), \quad D \equiv (\tfrac{1}{2}, 0).$$

Il est entendu que ces cas conventionnels de la notation (p, q) sont les seuls où les nombres figuratifs ne soient pas entiers. Ceci admis, on a l'énoncé suivant :

Théorème I. — *Soient cinq conditions élémentaires* (p, q), (p', q'), (p'', q''), (p''', q'''), $(p^{\text{iv}}, q^{\text{iv}})$, *rangées de telle sorte qu'on ait*

$$\frac{p}{q} < \frac{p'}{q'} < \frac{p''}{q''} < \frac{p'''}{q'''} < \frac{p^{\text{iv}}}{q^{\text{iv}}}.$$

Le nombre des coniques qui satisfont à ces cinq conditions est, dans tous les cas,

$$(1) \quad \ldots\ldots A = 8\,(2q+p)\,(2q'+p')\,(q''+p'')\,(q'''+2p''')\,(q^{\text{iv}}+2p^{\text{iv}}).$$

Pour être en mesure de démontrer ce résultat, je dois faire d'abord une étude des coniques dégénérées qui existent dans un système défini par quatre conditions élémentaires.

5. Soit, dans un système (a), une conique dégénérée suivant le troisième mode, c'est-à-dire ayant pour figure limite une droite C portant un point c. En dégénérant, la conique a intercepte sur une droite arbitraire Λ un segment infiniment petit d ; en outre, les tangentes qu'on peut lui mener d'un point arbitraire λ comprennent entre elles un angle infiniment petit δ. Les ordres de ces deux infiniment petits

d, δ sont entre eux dans le même rapport que l'ordre m et la classe n de la conique dégénérée (Mém. sur les Caract., No. 5). Il n'y a exception à cette règle que si Λ ou λ prennent des positions particulières relativement à la figure limite. Afin d'embrasser tous les cas, disons que d est infiniment petit d'ordre $m+\omega$ et δ d'ordre $n+\omega'$. Le nombre ω sera positif si la droite Λ contient c; il pourra être négatif dans le cas seulement où Λ se confondrait avec C. Le nombre ω' sera positif si la point λ est sur C; il pourra être négatif dans le cas seulement où λ se confondrait avec c.

Supposons maintenant que, parmi les conditions définissant le système (ω), figure la condition (p, q). Envisageons la droite L et le point l qui définissent (p, q), et considérons, pour cette droite et pour ce point, les quantités analogues à d, δ. Ces dernières sont du même ordre infinitésimal respectivement que les quantités r et R qui définissent (p, q) (No. 2). On devra donc avoir

$$\frac{m+\omega}{n+\omega'} = \frac{p}{q}.$$

De cette égalité résulte les conséquences suivantes: *Si, parmi les conditions qui définissent un système, figure la condition* (p, q) *et que ce système contienne une conique dégénérée d'ordre* m *et de classe* n, *ayant pour droite limite* C *et pour point limite* c;

1°. *Si le rapport* $h = m : n$ *surpasse* $p : q$, C *passe par* l *ou coïncide avec* L;

2°. *Si* h *est inférieur à* $p : q$, c *est sur* L *ou coïncide avec* l.

3°. *Si* C *ne passe pas par* l *et ne coïncide pas avec* L, *et que le point* C *ne soit pas sur* L *et ne coïncide pas avec* l, *le rapport* h *est égal à* $p : q$.

6. Dans cette analyse, j'ai considéré une condition (p, q) quelconque. Il est visible que les conclusions s'appliquent encore dans le cas où, au lieu de (p, q), on considère P ou D. En d'autres termes, le dernier énoncé s'applique aussi si l'un des nombres p, q est nul. Effectivement, dans le cas de la condition P, les coniques du système passent toutes par un même point l. Il en est donc de même de la droite C, limite de ces coniques. Ce fait est d'accord avec l'énoncé, puisque le rapport $p : q$ est égal à zéro et, par suite, inférieur à h. De même, dans le cas de la condition D, les coniques du système touchent toutes une même droite L; le point c est sur cette droite. Ce fait est d'accord avec l'énoncé, puisque le rapport $p : q$ est infini et, par suite, supérieur à h.

Il y a plus: en utilisant une convention déjà introduite (*loc. cit.*, No. 8), on peut étendre la proposition aux coniques dégénérées suivant le premier ou le second mode. C'est ce que je vais montrer.

D'après la convention dont il s'agit, le premier mode de dégénéres-

cence est caractérisé par l'ordre zéro. La figure limite se compose alors d'un point c, par où passent deux droites limites C_1, C_2. Supposons que cette dégénérescence existe dans un système défini par des conditions au nombre desquelles soit (p, q). En vertu de l'analyse précédente qui n'est pas troublée par l'hypothèse $m = 0$, on reconnait que le point c est sur la droite L, ou bien se confond avec le point l. Cette conclusion subsiste encore si (p, q) devient la condition D, ou condition de toucher une droite L. Il n'y a exception que s'il s'agit de la condition P. Mais cette condition est représentée par le symbole $(0, \frac{1}{2})$. Les rapports h et $p : q$ sont alors égaux, et cette circonstance n'infirme pas l'énoncé ci-dessus.

De même, le second mode de dégénérescence est caractérisé par la classe zéro. La figure limite se compose d'une droite C portant deux points c_1, c_2. Si cette dégénérescence existe dans un système de coniques satisfaisant toutes à (p, q), alors C passe par l ou se confond avec L, dans tous les cas sauf celui où (p, q) n'est autre que D. Mais alors les rapports h et $p : q$ sont tous deux infinis, et l'énoncé du No. 5 subsiste. Ainsi la proposition du No. 5 est entièrement générale.

7. J'en deduis maintenant cette conséquence: *Soient quatre conditions élémentaires* (p, q), (p', q'), (p'', q''), (p''', q'''), *rangées de telle sorte qu'on ait:*

$$\frac{p}{q} < \frac{p'}{q'} < \frac{p''}{q''} < \frac{p'''}{q'''}.$$

Dans le système défini par ces quatre conditions, supposées indépendantes, *pour toute conique dégénérée le rapport de l'ordre à la classe est l'un des deux rapports moyens* $p' : q'$ *ou* $p'' : q''$.

En effet, d'abord si le rapport $h = m : n$ est inférieur à trois des rapports $p : q$, chacune des trois conditions (p, q) correspondantes se traduit par une condition imposée au seul point limite c, ce qui ne se peut en vertu de l'hypothèse d'indépendance des conditions données (No. 3). De même h ne peut être supérieur à trois des rapports $p : q$. Car il en résulterait trois conditions pour la droite limite C. Donc h est nécessairement compris entre les deux rapports moyens. Par suite, tout d'abord h ne peut être nul que si $p' : q'$ l'est aussi, et h ne peut être infini que si $p'' : q''$ l'est également. Donc à l'égard des deux premiers modes de dégénérescence on obtient des conclusions conformes à l'énoncé, savoir: le premier mode $(h = 0)$ n'existe que si (p', q') coïncide avec P; le second mode $(h = \infty)$ n'existe que si (p'', q'') coïncide avec D.

A l'égard du troisième mode de dégénérescence, on remarquera que la figure limite Cc ne contient que trois arbitraires. Cette figure ne peut

donc satisfaire qu' à trois conditions provenant des conditions (p, q). Donc, d'après la proposition précédente, h coïncide avec un des rapports $p : q$. Mais, devant être compris entre les deux moyens, h est nécessairement égal à l'un ou l'autre de ces moyens. La proposition est donc pleinement prouvée.

8. Considérons toujours le même système. Il ne peut, comme on vient de le prouver, contenir que des coniques dégénérées de deux espèces. S'il contient effectivement la dégénérescence $h = p' : q'$, la somme des ordres et la somme des classes, pour toutes les coniques dégénérées de cette espèce, sont deux nombres proportionnels à p', q'. Ces sommes peuvent donc être représentées par $\omega' p'$ et $\omega' q'$. ω' étant un nombre inconnu qui serait zéro dans le cas de non-existence de ces coniques dégénérées. De même, par $\omega'' p''$ et $\omega'' q''$ nous représenterons la somme des ordres et la somme des classes pour la dégénérescence $h = p'' : q''$.

Soient μ, ν les caractéristiques du système. Désignons par M la somme des ordres, par N la somme des classes de toutes les coniques dégénérées. Entre ces nombres existent les relations (*loc. cit.*, No. 19)

$$2\mu - \nu = M, \quad 2\nu - \mu = N.$$

On a donc ici :

$$\omega' p' + \omega'' p'' = 2\mu - \nu, \quad \omega' q' + \omega'' q'' = 2\nu - \mu.$$

Si l'on connaît μ, ν, on aura là deux équations du 1er degré qui détermineront les inconnues ω', ω''. Il n'y aurait indétermination qu'au cas $p' : q' = p'' : q''$. Mais, les deux rapports moyens étant égaux, il ne peut exister alors qu'une seule espèce de dégénérescence ; il n'y a qu'une inconnue que l'une ou l'autre des équations déterminent.

Ainsi, pour les systèmes définis par quatre conditions élémentaires entendues comme je le fais ici, la connaissance des caractéristiques du système entraîne celle des caractéristiques des coniques dégénérées, tout aussi bien que pour les systèmes définis par les seules conditions P, D. Par exemple, admettons pour un instant l'exactitude du théorème I, que je vais bientôt démontrer ; nous pourrons en conclure les nombre ω', ω''. Pour trouver μ, il faudra appliquer ce théorème en envisageant les cinq conditions suivantes, rangées par ordre, $(0, \frac{1}{2})$, (p, q), (p', q'), (p'', q''), (p''', q'''). Il résulte alors de (1)

$$(2) \quad \ldots\ldots \mu = 8\,(2q+p)\,(q'+p')\,(q''+2p'')\,(q'''+2p''').$$

Pour trouver ν, on envisagera (p, q), (p', q'), (p'', q''), (p''', q''') $(\frac{1}{2}, 0)$, et la formule (1) donnera

$$(3) \quad \ldots\ldots \nu = 8\,(2q+p)\,(2q'+p')\,(q''+p'')\,(q'''+2p''').$$

Les formules (2) et (3) donnent aisément :

$$2\mu-\nu = 8(2q+p)(q''+2p'')(q'''+2p''')p' \\ +8(2q+p)(2q'+p')(q'''+2p''')p'',$$

$$2\nu-\mu = 8(2q+p)(q''+2p'')(q'''+2p''')q' \\ +8(2q+p)(2q'+p')(q'''+2p''')q''.$$

D'où je tire cette conclusion

$$(4) \ldots\ldots \omega' = 8(2q+p)(q''+2p'')(q'''+2p'''),$$

$$(5) \ldots\ldots \omega'' = 8(2q+p)(2q'+p')(q'''+2p''').$$

Ainsi, dès qu'on admet le théorème I. on a cette conséquence.

THÉORÈME II.—*Dans le système défini par quatre conditions élémentaires* (p, q), (p', q'), (p'', q''), (p''', q''') *telles que*

$$\frac{p}{q} < \frac{p'}{q'} < \frac{p''}{q''} < \frac{p'''}{q'''},$$

il existe des coniques dégénérées pour lesquelles le rapport de l'ordre à la classe est $p' : q'$. *La somme de leurs ordres est* $\omega' p'$, *le nombre* ω' *étant donné par l'égalité* (4). *Il y existe aussi des coniques pour lesquelles le rapport de l'ordre à la classe est* $p'' : q''$. *La somme de leurs ordres est* $\omega'' p''$, *le nombre* ω'' *étant donné par l'égalité* (5). *Les caractéristiques du système sont données par les égalités* (2) *et* (3).

REMARQUE.—*Dans le cas où* $p' : q' = p'' : q''$, *le rapport de l'ordre à la classe est* $p' : q' = p'' : q''$ *pour toutes les coniques dégénérées, et la somme des ordres est* $(\omega' + \omega'')p'$.

9. Ces conséquences tirées du théorème I. supposé exact vont précisément servir à prouver son exactitude. Je vais suivre, en effet, la marche suivante. J'admettrai le théorème pour le cas où l'une au moins des cinq conditions est P ou D, et je montrerai que, cette hypothèse admise, le théorème est encore vrai si ces conditions P, D figurent une fois de moins au nombre des données. Je vérifierai ensuite que le théorème est exact si toutes les cinq conditions sont P ou D. La démonstration sera ainsi faite. Or, si j'admets la formule (1) pour le cas où une des conditions est P ou D, j'admets alors que les caractéristiques μ, ν sont données par les formules (2), (3) pour un système élémentaire quelconque. Il en résulte alors l'exactitude des formules (4), (5). En d'autres termes, l'hypothèse introduite revient à supposer exact le théorème II., et il faut prouver que le théorème I. en est la conséquence. Connaissant les caractéristiques et les dégénérescences du système défini par quatre conditions élémentaires, je peux calculer le nombre des coniques de ce système qui satisfont à une cinquième condition. Mon mémoire précédent en fournit le moyen. Je suis donc

en mesure de démontrer le théorème I. par cette voie. Je rappelle la proposition qu'il faudra appliquer (*loc. cit.*, No. 28) :

Soit, dans un système dont les caractéristiques sont μ, ν, *une conique dégénérée d'ordre* m *et de classe* n. *Soit* γ *le plus petit des deux produits* mq, np, *et* Γ *la somme des nombres analogues à* γ *pour toutes les coniques dégénérées du système. Le nombre des coniques de ce système qui satisfont à la condition* (p, q) *est* $2(q\mu+p\nu)-\Gamma$.

10. Pour faire la démonstration que je viens d'indiquer, je mets à part une des cinq conditions données et je forme avec les autres un système. Il est nécessaire que la condition ainsi mise à part ne soit ni P ni D, sans quoi la démonstration serait illusoire. Si donc on veut qu'aucun cas ne soit omis, il faut distinguer autant de cas différents que la condition mise à part peut occuper de places différentes dans le classement des rapports $p : q$ par ordre de grandeur. Il y a donc cinq cas à distinguer. Mais on peut se borner à en prendre trois, eu égard a la dualité. Effectivement le principe de dualité s'applique, comme il convient à la formule (1). La condition (p, q) a pour corrélative (q, p). Si l'on tient compte de l'ordre, l'ensemble

$$(p, q),\ (p', q'),\ (p'', q''),\ (p''', q'''),\ (p^{\text{iv}}, q^{\text{iv}})$$

a pour corrélatif

$$(q^{\text{iv}}, p^{\text{iv}}),\ (q''', p'''),\ (q'', p''),\ (q', p'),\ (q, p),$$

et le nombre A n'est pas changé.

Il suffira donc de mettre à part, pour la démonstration, la condition qui occupe une quelconque des trois dernières places. Deux de ces cas d'ailleurs n'exigent ensemble qu'une seule démonstration. Il n'y a donc en réalité que deux démonstrations à faire.

1°. Je mets à part la condition $(p^{\text{iv}}, q^{\text{iv}})$ que je suppose être, à volonté, la quatrième ou la cinquième. Les quatre autres forment le système déjà envisagé. Les caractéristiques μ, ν sont connues par les formules (2) et (3). Pour calculer le nombre Γ (No. 9), supposons une conique dégénérée d'ordre m et de classe n ; le rapport $m : n$ est soit $p' : q'$, soit $p'' : q''$. Comme, par hypothèse, $p^{\text{iv}} : q^{\text{iv}}$ est supérieur à ces deux derniers rapports, le plus petit des deux produits mq^{iv} et np^{iv} est mq^{iv}. Donc Γ est la somme des nombres mq^{iv}, c'est-à-dire le produit de q^{iv} par la somme des ordres $\omega' p' + \omega'' p''$. Donc le nombre des coniques faisant partie du système et satisfaisant à $(p^{\text{iv}}, q^{\text{iv}})$, c'est-à-dire satisfaisant aux cinq conditions données, est

$$\begin{aligned} A &= 2q^{\text{iv}}\mu + 2p^{\text{iv}}\nu - (\omega' p' + \omega'' p'')\, q^{\text{iv}} \\ &= 2q^{\text{iv}}\mu + 2p^{\text{iv}}\nu - (2\mu - \nu)\, q^{\text{iv}} = (q^{\text{iv}} + 2p^{\text{iv}})\, \nu ; \end{aligned}$$

et si l'on met pour ν l'expression (3), on retrouve la formule (1).

2°. Je mets maintenant à part la condition moyenne (p'', q''). Le système défini par les quatre autres ne diffère de celui qu'on a considéré précédemment que par la notation : les accents $'''$ et $''$ y sont remplacés par ^{iv} et $'''$. Il n'y a donc à faire que ce changement dans les formules (2), (3), (4), (5). Pour les dégénérescences $h = p' : q'$, l'élément du nombre Γ est, comme tout-à-l'heure, $\omega' p' q''$, tandis que pour $h = p''' : q'''$, cet élément est $\omega''' q''' p''$. On a donc ici :

$$A = 2q''\mu + 2p''\nu - \omega' p' q'' - \omega''' q''' p'',$$

$$A = 8(2q+p)(q^{iv}+2p^{iv})\{2(q'+p')(q'''+2p''')q'' - (q'''+2p''')p'q''$$
$$+2(2q'+p')(q'''+p''')p'' - (2q'+p')q'''p''\},$$

$$A = 8(2q+p)(2q'+p')(q''+p'')(q'''+2p''')(q^{iv}+2p^{iv}),$$

ce qui est bien la formule (1).

11. Il reste enfin à vérifier l'exactitude de la formule pour le cas où les conditions sont toutes P ou D, ce qui fait six cas distincts renfermés dans le symbole $P^i D^{5-i}$, ce symbole signifiant que la conique doit passer par i points et toucher $(5-i)$ droites. A cause des propriétés dualistiques de la formule, il suffit de trois vérifications ; les voici :

1°. Conditions P^5.

$$p = p' = p'' = p''' = p^{iv} = 0, \quad q = q' = q'' = q''' = q^{iv} = \tfrac{1}{2}, \quad A = 1.$$

2°. Conditions $P^4 D$.

$$p = p' = p'' = p''' = 0, \quad p^{iv} = \tfrac{1}{2}, \quad q = q' = q'' = q''' = \tfrac{1}{2}, \quad q^{iv} = 0, \quad A = 2.$$

3°. Conditions $P^3 D^2$.

$$p = p' = p'' = 0, \quad p''' = p^{iv} = \tfrac{1}{2}, \quad q = q' = q'' = \tfrac{1}{2}, \quad q''' = q^{iv} = 0, \quad A = 4.$$

La vérification étant complète, le théorème I. est pleinement démontré, et le problème qui fait l'objet de ce mémoire est résolu.

12. Avec le théorème I. se trouve aussi démontré le théorème II., et l'on sait dorénavant calculer les dégénérescences qui existent dans un système défini par quatre conditions élémentaires. Il ne faut pas perdre de vue l'indépendance supposée des quatre conditions, sans laquelle le théorème II. pourrait être en défaut. Prenons pour exemple le système $P^2 D^2$, formé des coniques qui passent par deux points b, b' et touchent deux droites G, G'. Il y existe la dégénérescence d'ordre zéro, ayant pour classe totale le nombre 4, et aussi la dégénérescence de classe zéro, ayant pour ordre total le nombre 4. Mais, dans le cas particulier où la droite bb' concourt avec G et G', ces deux dégénérescences disparaissent et sont remplacées par la dégénérescence d'ordre total 2 et de classe totale 2, le rapport $m : n$ étant l'unité.

Connaissant les dégénérescences qui existent dans les systèmes

élémentaires, on les connaît aussi du même coup pour les systèmes définis par quatre conditions indépendantes quelconques. Car, à ce point de vue encore, chaque condition peut être évidemment remplacée par son équivalent. Inversement, il en résulte un procédé géométrique pour définir l'équivalent, tandis que jusqu' à présent il n'avait qu'une origine algébrique. C'est ce que je vais montrer.

13. Soit Φ une condition quelconque, et supposons que son équivalent contienne le terme $c\,(p,\ q)$. Envisageons le système $P^2\Phi D$. A notre point de vue, ce système est équivalent à l'ensemble ou *somme* de plusieurs systèmes élémentaires, parmi lesquels existe c fois le système $P^2\,(p,\ q)\ D$. Ce dernier contient la dégénérescence $h = p : q$, et d'après le théorème II., l'ordre total de cette dégénérescence est $8p$. Le même système contient, en outre, la dégénérescence d'ordre zéro, ayant pour classe totale $4\,(q+2p)$.

Inversement, si le système $P^2\Phi D$ contient la dégénérescence $h = p : q$, et que l'ordre total de cette dégénérescence soit $8cp$, on en conclura que l'équivalent de Φ contient le terme $c\,(p, q)$. Chaque espèce de dégénérescence du système fournit donc un terme de l'équivalent de Φ, et ceci s'applique même à la dégénérescence $h = \infty$. Si donc le système contient cette dégénérescence, et que l'ordre total correspondant soit $4b$, l'équivalent contient le terme $b\,(\frac{1}{2},\ 0)$ ou bD. Il n'y a d'exception à faire que pour les dégénérescences $h = 0$, qui sont introduites nécessairement par la présence de P comme condition moyenne. Mais, après avoir déterminé tous les termes de l'équivalent, sauf celui qui contient P, si l'on trouve que la dégénérescence d'ordre zéro existe avec la classe totale $4a'$, on en conclura le terme aP, le coefficient a étant défini par l'égalité :

$$a = a' - b - c\ (q+2p) - c'\ (q'+2p') \ \ldots\ldots$$

Ainsi *l'équivalent d'une condition quelconque peut être déterminé par l'étude géométrique du système composé des coniques satisfaisant à cette condition, passant par deux points et touchant une droite.* Je me borne ici à cette simple remarque dont l'application me détournerait du sujet actuel. D'ailleurs, comme je le montrerai dans une autre occasion, on peut déterminer l'équivalent au moyen de systèmes un peu plus simples. Mais je veux tirer une conséquence nouvelle et montrer qu'on possède maintenant le moyen de former à volonté, par des procédés purement géométriques, des conditions Φ dont les équivalents contiennent effectivement des termes (p, q).

14. Pour que l'équivalent de Φ contienne (p, q) il faut et il suffit que tout système $P^2\Phi D$ contienne la dégénérescence $h = p : q$. Mais les conditions P^2D déterminent, à elles seules, la figure limite Cc; la droite C joint les deux points donnés, le point c est l'intersection de C avec

la droite donnée. Ceci revient à dire que, dans l'ensemble triplement infini des systèmes de coniques satisfaisant à Φ, il existe une série triplement infinie de figures limites Cc correspondant à la dégénérescence $h = p : q$. Prenons donc un système contenant cette dégénérescence, et rendons libres trois éléments descriptifs de ce système, de telle sorte que la figure limite Cc contienne trois arbitraires; nous aurons ainsi défini une condition Φ contenant, dans son équivalent, le terme (p, q). C'est ce qu'on va bien comprendre par des exemples.

1er Exemple.—Soit la condition Φ suivante: *On donne deux droites B, B' et deux points g, g'. On astreint une conique à ce que la droite qui joint un de ses points b de rencontre avec B et un de ses points de rencontre b' avec B' concoure avec une des tangentes G qu'on peut lui mener de g et une des tangentes G' qu'on peut lui mener de g'.* Je dis que *cette condition Φ contient, dans son équivalent, le terme* (1, 1). Effectivement, prenons à volonté G et G' passant respectivement par g, g', et par leur rencontre menons arbitrairement une droite rencontrant B, B' en b, b'. Le système des coniques qui passent par b, b' et touchent G, G' contient la dégénérescence $h = 1$, comme on l'a déjà observé (No. 13). Toutes ces coniques satisfont à la condition Φ, et d'ailleurs, par construction, la figure limite peut être prise arbitrairement: la droite limite est bb', le point limite est le point de concours de G, G', et bb'. Donc l'équivalent de Φ contient le terme (1, 1) correspondant à la dégénérescence $h = 1$. Le coefficient de ce terme est à déterminer; mais cette détermination exigerait des développements auxquels je ne m'arrête pas ici.

2me Exemple.—Soit la condition suivante: *On donne une conique b et un point β sur cette courbe. On astreint une conique a à avoir un contact du 3me ordre avec une conique d indéterminée, mais ayant au point β, avec b, un contact du 2d ordre. Cette condition Φ, imposée à la conique a, contient aussi dans son équivalent un terme* (1, 1). En effet, si l'on prend arbitrairement un point c et une droite C passant par ce point, il existe une conique d touchant C en c, et ayant, en β avec b, un contact du 2d ordre. Le système des coniques a qui ont avec d, en c, des contacts du 3me ordre, contient la dégénérescence (1, 1) avec Cc pour figure limite. Toutes les coniques de ce système satisfont à Φ. Donc etc. A l'égard de cette condition, il est aisé de déterminer entièrement son équivalent, et de prouver que $\Phi \equiv (1, 1)$. Le moyen le plus simple est de former l'équation qui exprime Φ. Soit $2x_1x_3 + kx_2^2 = 0$ l'équation de b, et soit $x_1 = x_2 = 0$ le point β. Désignons par a_x^2 le premier membre de l'équation ponctuelle de a, par a_ξ^2 le premier membre de son équation tangentielle, par D le discriminant de a_x^2; l'équation de condition est $a_{11}^3 + k^2 Da_{33}^3 = 0$. Les principes de mon mémoire antérieur permettent de tirer immédiatement de cette équation l'équivalent de la condition.

3me Exemple. — *On astreint la conique a à avoir un contact du 4me ordre avec une cubique de 3me classe, indéterminée, mais assujettie à 4 conditions, comme de passer par deux points et de toucher deux droites. L'équivalent de cette condition contient le terme* (2, 1) *et le terme* (1, 2). En effet, prenons à volonté un point c et une droite C par ce point. Il existe une ou plusieurs cubiques K satisfaisant aux quatre conditions données et ayant c et C pour point et tangente d'inflexion. Considérons le système des coniques ayant des contacts du 4me ordre avec K. Dans ce système, la conique qui répond à l'inflexion c est dégénérée, son ordre est 4, sa classe est 2 ; le rapport h est égal à 2. Les coniques de ce système satisfont à la condition envisagée. Donc l'équivalent de cette condition contient le terme (2, 1). Si l'on substitue le rebroussement à l'inflexion, on verra aussi que l'équivalent contient le terme (1, 2).

D'une manière tout analogue on pourra, en substituant à la cubique d'autres courbes, définir pour les coniques des conditions géométriques dont les équivalents contiendront les termes (p, q) que l'on voudra.

15. Il me reste à montrer par une application comment le nombre A fourni par la formule (1) diffère de celui que l'on obtiendrait si l'on se servait à tort, dans certains cas, des formules de l'ancienne théorie. On voit aisément que la différence n'existe que si deux des conditions, au moins, ne sont ni P ni D, et si, en outre, les trois autres ne sont pas toutes soit P, soit D. L'exemple le plus simple est donc relatif aux conditions $P^2\,(p, q)\,(p', q')\,D$. La formule (1) donne alors :

$$A = 8\,(q+p)\,(q'+2p')$$

sous la condition $\qquad p : q \leqq p' : q'.$

L'ancienne théorie donnerait, sous forme symbolique :

$$A_1 = P^2D\,(2qP+2pD)\,(2q'P+2p'D) = 8\,[qq'+2\,(pq'+qp'+pp')].$$

Ce nombre A_1 surpasse le nombre exact A de $8q'p$. Pour prendre un exemple où ne figurent pas les conditions élémentaires, ayons recours à un des résultats de mon mémoire précédent :

On assujettit une conique a à ce que le rapport anharmonique de ses points d'intersection avec une conique donnée b, pris sur a dans un ordre quelconque, soit inverse du rapport anharmonique de ces mêmes points pris, dans le même ordre, sur b. On assujettit a à cette même condition relativement à une seconde conique donnée C ; et, en outre, à passer par deux points donnés et à toucher une droite donnée. Combien de coniques a répondent à la question ?

Les deux premières conditions ont chacune pour équivalent (1, 1). Il n'y a donc qu'à appliquer la dernière formule avec $p=p'=q=q'=1$. Donc *il y a* 48 *coniques a*, tandis que l'ancienne théorie, appliquée ici à

tort, en donnerait 56. Si, au lieu de deux conditions (p, q), on en introduit un plus grand nombre, l'écart entre les deux résultats ira en grandissant. Par exemple, pour cinq conditions (1, 1) le nombre exact est $C^4 = 2376$, tandis que l'ancienne théorie donnerait le nombre 3264, le même que pour celui des coniques qui touchent cinq coniques données.

Note on a Modular Equation for the Transformation of the Third Order. *By* Henry J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

[*Read February 13th*, 1879.]

I have given elsewhere* the modular equation for the transformation of the third order between

$$x = f(k^2) = \frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2} \quad \text{and} \quad y = f(\lambda^2) = \frac{(1-\lambda^2+\lambda^4)^3}{\lambda^4(1-\lambda^2)^2};$$

viz.,
$$\begin{aligned} F(x, y) = {} & x(x+2^7.3.5^3)^3 + y(y+2^7.3.5^3)^3 \\ & -2^{16}x^3y^3 + 2^{11}.3^2.31\,x^2y^2(x+y) \\ & -2^2.3^3.9907\,xy(x^2+y^2) + 2.3^4.13.193.6367\,x^2y^2 \\ & +2^8.3^5.5^3.4471\,xy(x+y) - 2^{15}.5^6.22973\,xy = 0. \end{aligned}$$

The following is the process by which the coefficients were determined.

It follows from the general theory of modular equations that $F(x, y)$ is symmetrical with respect to x and y; and that it is of the order 4 in x and y separately, and of the order 6 in x and y jointly. Hence $F(x, y)$ is of the form

$$\begin{aligned} & A_{3,3}x^3y^3 + A_{4,2}x^2y^2(x^2+y^2) + A_{4,1}xy(x^3+y^3) \\ & \quad + (x^4+y^4) + A_{3,2}x^2y^2(x+y) + A_{3,1}xy(x^2+y^2) \\ & \quad + A_{3,0}(x^3+y^3) + A_{2,2}x^2y^2 + A_{2,1}xy(x+y) \\ & \quad + A_{2,0}(x^2+y^2) + A_{1,1}xy + A_{1,0}(x+y) + A_{0,0}. \end{aligned}$$

Several of the coefficients may be conveniently found by employing the method of Sohncke; *i.e.*, by substituting for x and y their expressions as series proceeding by powers of q, and equating the coefficients of the powers of q to zero. We have, by a known formula,

$$k^2(1-k^2) = 2^4 q \prod_{m=1}^{m=\infty} (1+q^{2m-1})^{-24};$$

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* "Proceedings of the London Mathematical Society," Vol. IX., p. 243, note.

calling this quantity z, we have $x = \frac{(1-z)^3}{z^2}$;

we also write $$Z = 2^4 q^3 \prod_{m=1}^{m=\infty} (1+q^{6m-3})^{-24},$$

so that $$y = \frac{(1-Z)^3}{Z^2}.$$

Substituting these values, and equating to zero the coefficients of q^{-28}, q^{-26}, and q^{-24}, we find successively

$$A_{4,2} = 0, \quad A_{4,1} = 0, \quad A_{3,3} = -2^{16}.$$

To determine $A_{3,2}$, we observe that q^{-22} presents itself only in the terms

$$-2^{16} x^5 y^3 + A_{3,2} x^2 y^3.$$

Its coefficient in x^2y^3 is 2^{-40}; its coefficient in x^3y^3 is $3^2 . 31 \times 2^{-45}$; viz., this is the coefficient of q^2 in

$$\frac{1}{2^{24}} \left[\frac{1}{2^{24}} (1+q)^{144} - \frac{9}{2^{20}} q (1+q)^{120} + \frac{9}{2^{14}} q^2 \right].$$

Hence $$A_{3,2} \times 2^{-40} + 2^{-45} . 3^2 . 31 . A_{3,3} = 0,$$

or $$A_{3,2} = 2^{11} . 3^2 . 31.$$

Again, to determine $A_{3,1}$ we consider the coefficient of q^{-20}; this power of q presents itself only in the terms

$$y^3 [A_{3,3} x^3 + A_{3,2} x^2 + A_{3,1} x].$$

It will be found that in the development of y^3 there is no power of q intermediate between q^{-18} and q^{-12}; hence, the coefficient of q^{-2} in

$$A_{3,3} x^3 + A_{3,2} x^2 + A_{3,1} x,$$

or the coefficient of q^4 in

$$\begin{aligned} A_{3,3} \Big[& \frac{1}{2^4} (1+q)^{144} (1+q^3)^{144} - \frac{3^2}{2^{20}} q (1+q)^{120} (1+q^3)^{120} \\ & + \frac{3^2}{2^{14}} q^2 (1+q)^{96} - \frac{3 . 7}{2^{10}} q^3 (1+q)^{72} - \frac{3^2 . 7}{2^7} q^4 \Big] \\ & + A_{3,2} \left[\frac{1}{2^{16}} q^2 (1+q)^{96} - \frac{3}{2^{11}} q^3 (1+q)^{72} + \frac{3 . 5}{2^8} q^4 \right] \\ & + \frac{A_{3,1}}{2^8} \times q^4 \end{aligned}$$

is equal to zero. Substituting the values of $A_{3,3}$, $A_{3,2}$, and reducing, we find $$A_{3,1} = -2^2 . 3^3 . 9907.$$

All the coefficients may successively be determined by this method, but the work becomes very laborious for the later coefficients. They

may be more easily obtained by the consideration that, if ξ, η are any two corresponding values of k^2 and λ^2, $f(\xi)$ and $f(\eta)$ are corresponding values of x and y. Availing ourselves of this principle, we find

$$\text{(i.)}\quad F(0, y) = y(y+2^7 \cdot 3 \cdot 5^3)^3,$$

$$\text{(ii.)}\quad F\left(\frac{27}{4}, y\right) = \left(y^2 - \frac{3^2}{2} \times 133283y - \frac{3^3 \cdot 11^3 \cdot 23^3}{2^4}\right)^2,$$

$$\text{(iii.)}\quad F(y, y) = -2^{16} y \left(y - \frac{3^3 \cdot 5^3}{16}\right)\left(y - \frac{5^3}{4}\right)^2 (y+2^7)^2.$$

From (1) we infer that $F(x, y)$ is of the form

$$\text{(A)}\ \ldots\ldots \begin{cases} y(y+2^7 \cdot 3 \cdot 5^3)^3 + x(x+2^7 \cdot 3 \cdot 5^3)^3 - 2^{16} x^3 y^3 \\ \quad + 2^{11} \cdot 3^2 \cdot 31 x^2 y^2 (x+y) - 2^2 \cdot 3^3 \cdot 9907 xy (x^2+y^2) \\ \quad + A_{2,2} x^2 y^2 + A_{2,1} xy (x+y) + A_{1,1} xy = 0. \end{cases}$$

From (iii.) we find

$$\begin{aligned} A_{1,1} &= -2^{15} \cdot 5^6 \cdot 22973, \\ A_{2,1} &= 2^8 \cdot 3^5 \cdot 5^3 \cdot 4471, \\ A_{2,2} &= 2^{11} \cdot 1262587 + 2^3 \cdot 3^3 \cdot 9907 - 2 \\ &= 2 \cdot 3^4 \cdot 15974803 \\ &= 2 \cdot 3^4 \cdot 13 \cdot 193 \cdot 6367. \end{aligned}$$

This completes the determination of the coefficients; a verification is supplied by the equation (ii.); and it only remains to show how the equations (i.), (ii.), (iii.) are inferred from the modular equation between $\xi = k^2$ and $\eta = \lambda^2$; viz.,

$$\Phi(\xi, \eta) = (\xi^2 + 6\xi\eta + \eta^2)^2 - 16\xi\eta(4\xi\eta - 3\xi - 3\eta + 4)^2 = 0.$$

(i.) Let ρ denote either root of the equation $\rho^2 - \rho + 1 = 0$; if $k^2 = \rho$, we have $x = f(\rho) = 0$; we also find

$$\Phi(\rho, \eta) = (\eta - \rho)(\eta^3 - [128 - 253\rho]\eta^2 - (128 + 253\rho^2)\eta + 1).$$

If ω_1, ω_2, ω_3 are the roots of the cubic factor, the four roots of $F(0, y)$ are $\qquad f(\rho) = 0, \quad f(\omega_1), \quad f(\omega_2), \quad f(\omega_3).$

But the cubic factor is one of the two conjugate factors of the expression $\qquad (\eta^2 - \eta + 1)^3 + 2^7 \cdot 3 \cdot 5^3 \cdot \eta^2 (\eta - 1)^2.$

Hence $\qquad f(\omega_1) = f(\omega_2) = f(\omega_3) = -2^7 \cdot 3 \cdot 5^3,$

and $\qquad F(0, y) = y(y+2^7 \cdot 3 \cdot 5^3)^3.$

(ii.) Let $k^2 = \frac{1}{2}$, so that $f(k^2) = \frac{27}{4}$; we find

$$\Phi\left(\tfrac{1}{2}, y\right) = (\eta - \eta^2)^2 - 48\tfrac{1}{2}(\eta - \eta^2) + \tfrac{1}{16}.$$

If $\eta_1, \eta_2, \eta_3, \eta_4$ are the roots of $\Phi\,(\frac{1}{2}, \eta)$, the four roots of $F\,(\frac{2\,7}{4}, y)$ are $f(\eta_1), f(\eta_2), f(\eta_3), f(\eta_4)$; *i.e.* if z_1, z_2 are the roots of $z^2 - 48\frac{1}{2}\,z + \frac{1}{1\,6}$, these four roots are $z_1^{-2}\,(1-z_1)^3$, $z_2^{-2}\,(1-z_2)^3$, each taken twice. Hence

$$F\,(\tfrac{2\,7}{4}, y) = [y - z_1^{-2}\,(1-z_1)^3]^2\,[y - z_2^{-2}\,(1-z_2)^3]^2\,;$$

and by the ordinary methods we find

$$[y - z_1^{-2}\,(1-z_1)^3]\,[y - z_2^{-2}\,(1-z_2)^3]^2$$
$$= y^2 - \frac{3^2}{2} \times 133283\,y - \frac{3^3 \cdot 11^3 \cdot 23^3}{2^4}.$$

(iii.) Since the only roots of the equation

$$f\,(\eta) = f\,(\xi)$$

are

$$\xi,\ 1-\xi,\ \frac{1}{\xi},\ \frac{1}{1-\xi},\ \frac{\xi-1}{\xi},\ \frac{\xi}{\xi-1},$$

the roots of the equation $F\,(y, y) = 0$ are all of the form $f(\theta)$, where θ is a root of one of the six equations—

(1) ..... $\Phi\,(\theta, \theta) = \theta^2\,(\theta-1)^2\,(\theta^2-\theta+1) = 0,$

(2) ..... $\Phi\,(\theta, 1-\theta) = (16\theta^2-16\theta+1)\,(4\theta^2-4\theta-1)^2 = 0,$

(3) ..... $\theta^4\,\Phi\left(\theta, \frac{1}{\theta}\right) = (\theta-1)^2\,(\theta^2-6\theta+1)^2\,(\theta^2+14\theta+1) = 0,$

(4) ..... $(1-\theta)^4\,\Phi\left(\theta, \frac{1}{1-\theta}\right)$
$= (\theta^2-\theta+1) \times [(1-\theta+\theta^2)^3 + 128\theta^2\,(1-\theta)^2] = 0,$

(5) ..... $\theta^4\,\Phi\left(\theta, \frac{\theta-1}{\theta}\right)$
$= (\theta^2-\theta+1) \times [(\theta^2-\theta+1)^3 + 128\theta^2\,(\theta-1)^2] = 0,$

(6) ..... $(\theta-1)^4\,\Phi\left(\theta, \frac{\theta}{\theta-1}\right) = \theta^2\,(\theta^2-16\theta+16)\,(\theta^2+4\theta-4)^2 = 0.$

These equations are of order 8, the first and second having each two infinite roots. The 48 roots give, in all, five distinct values for $f\,(\theta)$ according to the following scheme :—

A. $f\,(\theta) = \infty$; 12 roots, viz.—

$\theta = \infty$, 2 roots in (1) and in (2) ;
$\theta = 0$, 2 roots in (1) and in (6) ;
$\theta = 1$, 2 roots in (1) and in (3).

B. $f\,(\theta) = 0$; 6 roots, viz,

$\theta^2 - \theta + 1 = 0$, in (1), (4), and (5).

C. $f(\theta) = \frac{3^3 \cdot 5^3}{2^4}$; 6 roots, viz.—

$$16\theta^2 - 16\theta + 1 = 0, \text{ in (2)};$$
$$\theta^2 + 14\theta + 1 = 0, \text{ in (3)};$$
$$\theta^2 - 16\theta + 16 = 0, \text{ in (6)}.$$

D. $f(\theta) = -128$; 12 roots, viz.—

$$(\theta^2 - \theta + 1)^3 + 128\theta^2(\theta - 1)^2 = 0, \text{ in (4) and (5)}.$$

E. $f(\theta) = \frac{5^3}{2^2}$; 12 roots, viz.—

$$(4\theta^2 - 4\theta - 1)^2 = 0, \text{ in (2)};$$
$$(\theta^2 - 6\theta + 1)^2 = 0, \text{ in (3)};$$
$$(\theta^2 + 4\theta - 4)^2 = 0, \text{ in (6)}.$$

Hence the roots of $F(y, y) = 0$ are $0, \frac{3^3 \cdot 5^3}{2^4}$, each once; ∞, -2^7, $\frac{5^3}{2^2}$, each twice; *i.e.*—

$$F(y, y) = -2^{16} y \left(y - \frac{3^3 \cdot 5^3}{2^4}\right) \left(y - \frac{5^3}{2^2}\right)^2 (y + 2^7)^2.$$

The multiplicity of the roots of $F(y, y) = 0$ may be otherwise, and more simply, determined as follows. The form of $F(x, y)$ (see the equation A, *supra*) shows that $F(y, y)$ has one root equal to zero and two roots equal to infinity; the multiplicities of the finite roots are determined by the equation

$$2^{16}\left[\frac{3^3 \cdot 5^3}{2^4} + 2 \times \frac{5^3}{2^2} - 2 \times 2^7\right] = 2A_{3,2} = 2^{12} \cdot 3^2 \cdot 31.$$

Note on the Formula for the Multiplication of Four Theta Functions. *By* Henry J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

[*Read February 13th*, 1879.]

The normal formula for the multiplication of four Theta Functions is ("Proceedings of the London Mathematical Society," Vol. I., Part viii., p. 4)

$$(1) \ldots \begin{cases} 2\theta_{\mu_1,\mu_1'}(x_1)\,\theta_{\mu_2,\mu_2'}(x_2)\,\theta_{\mu_3,\mu_3'}(x_3)\,\theta_{\mu_4,\mu_4'}(x_4) \\ = \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j,\sigma'-\mu_j'}(s-x_j) + \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j,\sigma'-\mu_j'+1}(s-x_j) \\ + (-1)^{\sigma'} \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j+1,\sigma'-\mu_j'}(s-x_j) + (-1)^{\sigma'+1} \prod_{j=1}^{j=4} \theta_{\sigma-\mu_j+1,\sigma'-\mu_j'+1}(s-x_j). \end{cases}$$

In this formula we have

$$(2)\ \ldots\ldots \begin{cases} \theta_{\mu,\mu'}(x) = \sum_{m=-\infty}^{m=+\infty} (-1)^{m\mu'} q^{\frac{1}{2}(2m+\mu)^2} \epsilon^{i(2m+\mu)x} \\ = \sum_{m=-\infty}^{m=+\infty} (-1)^{m\mu'} \epsilon^{\frac{1}{2}i\pi\omega(2m+\mu)^2} \epsilon^{i(2m+\mu)x}; \end{cases}$$

$$(3)\ \ldots\ldots \begin{cases} 2s = x_1+x_2+x_3+x_4, \\ 2\sigma = \mu_1+\mu_2+\mu_3+\mu_4, \\ 2\sigma' = \mu_1'+\mu_2'+\mu_3'+\mu_4'; \end{cases}$$

$q = \epsilon^{i\pi\omega}$ being a constant of which the analytical modulus is inferior to unity, and the indices μ, μ' being integral numbers, which render the sums 2σ, $2\sigma'$ even.

I. *The Eleven Cases of the Formula.*

Since

$$(4)\ \ldots\ldots \begin{cases} \theta_{\mu+2,\mu'}(x) = (-1)^{\mu'}\theta_{\mu,\mu'}(x), \\ \theta_{\mu,\mu'+2}(x) = \theta_{\mu,\mu'}(x), \end{cases}$$

we need only attribute the values 1, 0 to the indices μ, μ'. We may also permute the four arguments x_1, x_2, x_3, x_4 in any way we please; thus the matrix $\left|\begin{matrix}\mu_1, & \mu_2, & \mu_3, & \mu_4 \\ \mu_1', & \mu_2', & \mu_3', & \mu_4'\end{matrix}\right|$ may have eleven different values, which are enumerated in the following table:—

Table I.

A. $\sigma \equiv 0$, $\sigma' \equiv 0$, mod. 2: Cases i.—iv.

$$\left|\begin{matrix}1&1&1&1\\1&1&1&1\end{matrix}\right|, \quad \left|\begin{matrix}1&1&1&1\\0&0&0&0\end{matrix}\right|, \quad \left|\begin{matrix}0&0&0&0\\1&1&1&1\end{matrix}\right|, \quad \left|\begin{matrix}0&0&0&0\\0&0&0&0\end{matrix}\right|.$$

B. $\sigma \equiv 0$, $\sigma' \equiv 1$, mod. 2: Cases v., vi.

$$\left|\begin{matrix}1&1&1&1\\1&1&0&0\end{matrix}\right|, \quad \left|\begin{matrix}0&0&0&0\\1&1&0&0\end{matrix}\right|.$$

C. $\sigma \equiv 1$, $\sigma' \equiv 0$, mod. 2: Cases vii., viii.

$$\left|\begin{matrix}1&1&0&0\\1&1&1&1\end{matrix}\right|, \quad \left|\begin{matrix}1&1&0&0\\0&0&0&0\end{matrix}\right|.$$

D. $\sigma \equiv 1$, $\sigma' \equiv 1$, mod. 2: Cases ix.—xi.

$$\left|\begin{matrix}1&1&0&0\\1&1&0&0\end{matrix}\right|, \quad \left|\begin{matrix}1&1&0&0\\1&0&1&0\end{matrix}\right|, \quad \left|\begin{matrix}1&1&0&0\\0&0&1&1\end{matrix}\right|.$$

The formulæ appertaining to these eleven cases are all different from one another; *i.e.*, none of them can be derived from any other by a permutation of the four arguments x; we can, however, pass from any one of them to any other by means of the formula which connects any two different Theta functions; viz.,

$$(5)\ \ldots\ldots \theta_{\mu+\nu,\mu'+\nu'}(x) = \epsilon^{i\pi\left[\nu\frac{x}{\pi}+\nu^2\frac{\omega}{4}-\frac{\mu\nu}{2}\right]} \times \theta_{\mu,\mu'}\left[x+\frac{\pi}{2}(\nu\omega+\nu')\right].$$

Using a notation with a single suffix, and writing

$$\theta_{0,1}(x) = \vartheta_0(x), \quad \theta_{1,1}(x) = \vartheta_1(x),$$
$$\theta_{1,0}(x) = \vartheta_2(x), \quad \theta_{0,0}(x) = \vartheta_3(x),$$

we may exhibit the eleven formulæ as follows.

TABLE II. *Formulæ for the Multiplication of Four Theta Functions.*

A.

(i.) $$2\vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 = \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 + \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 + \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 - \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3.$$

(ii.) $$2\vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 = \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 + \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 + \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 - \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0.$$

(iii.) $$2\vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 + \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 + \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 - \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2.$$

(iv.) $$2\vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 + \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 + \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 - \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1.$$

B.

(v.) $$2\vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2 = \vartheta_2\times\vartheta_2\times\vartheta_1\times\vartheta_1 + \vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2 - \vartheta_3\times\vartheta_3\times\vartheta_0\times\vartheta_0 + \vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3.$$

(vi.) $$2\vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_0\times\vartheta_0 + \vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3 - \vartheta_2\times\vartheta_2\times\vartheta_1\times\vartheta_1 + \vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2.$$

C.

(vii.) $$2\vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_1\times\vartheta_1 + \vartheta_3\times\vartheta_3\times\vartheta_2\times\vartheta_2 + \vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0 - \vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3.$$

(viii.) $$2\vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_2\times\vartheta_2 + \vartheta_0\times\vartheta_0\times\vartheta_1\times\vartheta_1 + \vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3 - \vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0.$$

D.

$$\begin{aligned}(\text{ix.})\quad & 2\vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3 = \\ & \vartheta_3\times\vartheta_3\times\vartheta_1\times\vartheta_1 + \vartheta_0\times\vartheta_0\times\vartheta_2\times\vartheta_2 \\ & - \vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0 + \vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3.\end{aligned}$$

$$\begin{aligned}(\text{x.})\quad & 2\vartheta_1\times\vartheta_2\times\vartheta_0\times\vartheta_3 = \\ & \vartheta_3\times\vartheta_0\times\vartheta_2\times\vartheta_1 + \vartheta_0\times\vartheta_3\times\vartheta_1\times\vartheta_2 \\ & + \vartheta_2\times\vartheta_1\times\vartheta_3\times\vartheta_0 - \vartheta_1\times\vartheta_2\times\vartheta_0\times\vartheta_3.\end{aligned}$$

$$\begin{aligned}(\text{xi.})\quad & 2\vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0 = \\ & \vartheta_0\times\vartheta_0\times\vartheta_2\times\vartheta_2 + \vartheta_3\times\vartheta_3\times\vartheta_1\times\vartheta_1 \\ & - \vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3 + \vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0.\end{aligned}$$

For brevity, the arguments x_1, x_2, x_3, x_4, and the arguments $s-x_1$, $s-x_2$, $s-x_3$, $s-x_4$ are omitted in the left and right hand members respectively.

It will be observed that of the 256 sets of values which may be attributed to the constituents of the matrix

$$\left|\begin{matrix}\mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4\end{matrix}\right|,$$

192 are excluded by the condition that 2σ and $2\sigma'$ are even; of the remaining 64, 4 are represented by the formulæ i.—iv., which are symmetrical with respect to all the arguments; 24 by the formula x., which is entirely unsymmetrical; and 6 by each of the formulæ v.—ix. and xi., which are symmetrical with respect to the arguments taken in pairs.

II. *Application of the Formula to the Abelian Functions.*

The Abelian functions are defined by the equations

$$(6)\ \ldots\ldots\begin{cases} Al_1(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\frac{\pi}{2K}\vartheta_1'(0)}, \\ Al_2(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_2\left(\frac{\pi x}{2K}\right)}{\vartheta_2(0)}, \\ Al_0(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)}, \\ Al_3(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_3\left(\frac{\pi x}{2K}\right)}{\vartheta_3(0)}; \end{cases}$$

the constants K and ζ being determined by the equations

$$(7) \ldots\ldots \sqrt{\frac{2K}{\pi}} = 1+2q+2q^4+2q^9+2q^{16}+\ldots,$$

$$(8) \ldots\ldots \tfrac{1}{4}\zeta = \frac{q-4q^4+9q^9-16q^{16}+\ldots}{1-2q+2q^4-2q^9+2q^{16}-\ldots}.$$

The formulæ of Table II. give rise to a corresponding system of formulæ for the multiplication of four Abelian functions. To obtain this second system, we have only to express the Theta functions as Abelian functions, and to attend to the equations

$$(9) \ldots\ldots \begin{cases} k^2 = \dfrac{\vartheta_2^4(0)}{\vartheta_3^4(0)}, \quad k'^2 = \dfrac{\vartheta_0^4(0)}{\vartheta_3^4(0)}, \\ \dfrac{\pi}{2K}\vartheta'_{1,1}(0) = i\,\dfrac{\vartheta_0(0)\,\vartheta_2(0)}{\vartheta_3(0)}; \end{cases}$$

TABLE III. *Formulæ for the Multiplication of Four Abelian Functions.*

A.

(i.) $$2Al_1\times Al_1\times Al_1\times Al_1 =$$
$$Al_1\times Al_1\times Al_1\times Al_1 + \frac{1}{k'^2}\,Al_2\times Al_2\times Al_2\times Al_2$$
$$+\frac{1}{k^2}Al_0\times Al_0\times Al_0\times Al_0 - \frac{1}{k^2k'^2}Al_3\times Al_3\times Al_3\times Al_3.$$

(ii.) $$2Al_2\times Al_2\times Al_2\times Al_2 =$$
$$Al_2\times Al_2\times Al_2\times Al_2 + k'^2\,Al_1\times Al_1\times Al_1\times Al_1$$
$$+\frac{1}{k^2}\,Al_3\times Al_3\times Al_3\times Al_3 - \frac{k'^2}{k^2}\,Al_0\times Al_0\times Al_0\times Al_0.$$

(iii.) $$2Al_0\times Al_0\times Al_0\times Al_0 =$$
$$Al_0\times Al_0\times Al_0\times Al_0 + \frac{1}{k'^2}\,Al_3\times Al_3\times Al_3\times Al_3$$
$$+k^2\,Al_1\times Al_1\times Al_1\times Al_1 - \frac{k^2}{k'^2}\,Al_2\times Al_2\times Al_2\times Al_2.$$

(iv.) $$2Al_3\times Al_3\times Al_3\times Al_3 =$$
$$Al_3\times Al_3\times Al_3\times Al_3 + k'^2\,Al_0\times Al_0\times Al_0\times Al_0$$
$$+k^2\,Al_2\times Al_2\times Al_2\times Al_2 - k^2k'^2\,Al_1\times Al_1\times Al_1\times Al_1.$$

B.

(v.) $$2Al_1\times Al_1\times Al_2\times Al_2 = Al_2\times Al_2\times Al_1\times Al_1 + Al_1\times Al_1\times Al_2\times Al_2 + \frac{1}{k^2}Al_3\times Al_3\times Al_0\times Al_0 - \frac{1}{k^2}Al_0\times Al_0\times Al_3\times Al_3.$$

(vi.) $$2Al_0\times Al_0\times Al_3\times Al_3 = Al_3\times Al_3\times Al_0\times Al_0 + Al_0\times Al_0\times Al_3\times Al_3 + k^2Al_2\times Al_2\times Al_1\times Al_1 - k^2Al_1\times Al_1\times Al_2\times Al_2.$$

C.

(vii.) $$2Al_1\times Al_1\times Al_0\times Al_0 = Al_0\times Al_0\times Al_1\times Al_1 - \frac{1}{k'^2}Al_3\times Al_3\times Al_2\times Al_2 + Al_1\times Al_1\times Al_0\times Al_0 + \frac{1}{k'^2}Al_2\times Al_2\times Al_3\times Al_3.$$

(viii.) $$2Al_2\times Al_2\times Al_3\times Al_3 = Al_3\times Al_3\times Al_2\times Al_2 - k'^2Al_0\times Al_0\times Al_1\times Al_1 + Al_2\times Al_2\times Al_3\times Al_3 + k'^2Al_1\times Al_1\times Al_0\times Al_0.$$

D.

(ix.) $$2Al_1\times Al_1\times Al_3\times Al_3 = + Al_3\times Al_3\times Al_1\times Al_1 - Al_0\times Al_0\times Al_2\times Al_2 + Al_2\times Al_2\times Al_0\times Al_0 + Al_1\times Al_1\times Al_3\times Al_3.$$

(x.) $$2Al_1\times Al_2\times Al_0\times Al_3 = Al_3\times Al_0\times Al_2\times Al_1 + Al_0\times Al_3\times Al_1\times Al_2 + Al_2\times Al_1\times Al_3\times Al_0 - Al_1\times Al_2\times Al_0\times Al_3$$

(xi.) $$2Al_2\times Al_2\times Al_0\times Al_0 = Al_0\times Al_0\times Al_2\times Al_2 - Al_3\times Al_3\times Al_1\times Al_1 + Al_1\times Al_1\times Al_3\times Al_3 + Al_2\times Al_2\times Al_0\times Al_0.$$

III. *Case when the Sum of the Four Arguments is zero.*

Putting $$s = x_1+x_2+x_3+x_4 = 0,$$

and attending to the equation

$$(10)\ \ldots\ldots\ \theta_{\mu,\mu'}(-x) = (-1)^{\mu\mu'}\theta_{\mu,\mu'}(x),$$

we find that in each of the formulæ i.—xi., one of the terms on the right-hand side cancels one of the two equal terms on the left, and that the four formulæ A., and the formulæ of the three pairs B., C., D., ix. and xi., become respectively coincident; the formula D. x. remains

sui generis. Introducing the elliptic functions

$$\operatorname{sn} x = \frac{Al_1(x)}{Al(x)}, \quad \operatorname{cn} x = \frac{Al_2(x)}{Al(x)}, \quad \operatorname{dn} x = \frac{Al_3(x)}{Al(x)}$$

into the five formulæ thus obtained, we arrive at the following system:—

(i.) $k^2k'^2\Pi \cdot \operatorname{sn} x - k^2\Pi \cdot \operatorname{cn} x + \Pi \cdot \operatorname{dn} x - k'^2 = 0,$

(ii.) $k^2 \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{cn} x_3 \operatorname{cn} x_4 - k^2 \operatorname{cn} x_1 \operatorname{cn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4 - \operatorname{dn} x_1 \operatorname{dn} x_2 + \operatorname{dn} x_3 \operatorname{dn} x_4 = 0,$

(iii.) $k'^2 \operatorname{sn} x_1 \operatorname{sn} x_2 - k'^2 \operatorname{sn} x_3 \operatorname{sn} x_4 + \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{cn} x_3 \operatorname{cn} x_4 - \operatorname{cn} x_1 \operatorname{cn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4 = 0,$

(iv.) $\operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4 - \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4 + \operatorname{cn} x_3 \operatorname{cn} x_4 - \operatorname{cn} x_1 \operatorname{cn} x_2 = 0,$

(v.) $\operatorname{sn} x_1 \operatorname{cn} x_2 \operatorname{dn} x_4 + \operatorname{dn} x_1 \operatorname{cn} x_3 \operatorname{sn} x_4 + \operatorname{dn} x_2 \operatorname{sn} x_3 \operatorname{cn} x_4 + \operatorname{cn} x_1 \operatorname{sn} x_2 \operatorname{dn} x_3 = 0.$

The first of these, which alone is symmetrical with respect to the four arguments, is the formula given by Professor Cayley ("Proceedings," p. 43). The formulæ (ii.), (iii.), (iv.) are symmetrical with respect to the two pairs x_1x_2, x_3x_4, and with respect to the two arguments of each pair; thus each of these formulæ represents a set of three. Lastly, the formula (v.) remains unchanged when any two of the arguments are interchanged, provided that the other two are interchanged at the same time; *i.e.*, there are six formulæ of this type.

As a verification of these formulæ in a particular case, let $x_4 = 0$; we find

$$(11) \ldots\ldots \begin{cases} Dd_1d_2 - k^2Cc_1c_2 = k'^2, \\ D + k^2Cs_1s_2 = d_1d_2, \\ Dc_1c_2 - Cd_1d_2 = k'^2s_1s_2, \\ C + Ds_1s_2 = c_1c_2, \\ Dc_1s_2 - Sd_2 = -s_1c_2, \end{cases}$$

(where we have written, for brevity, $s_1 = \operatorname{sn} x_1$, $c_1 = \operatorname{cn} x_1$, $d_1 = \operatorname{dn} x_1$; $s_2c_2d_2$ and SCD having similar meanings with respect to the arguments x_2 and $x_1 + x_2 = -x_3$); and these equations are easily shown to be true by means of the formulæ for the addition of elliptic functions.

IV. *Formula for the Multiplication of Four Multiple Theta Functions.*

We define a double Theta function by the equation

$$(12)\ \ldots\ldots \begin{cases} \theta(\mu, \mu';\ \nu, \nu';\ x, y) = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} (-1)^{m\mu'-n\nu} \\ \times A^{\frac{1}{4}(2m+\mu)^2} \times B^{\frac{1}{2}(2m+\mu)(2n+\nu)} \times C^{\frac{1}{4}(2n+\nu)^2} \\ \times \epsilon^{(2m+\mu)ix+(2n+\nu)iy} \\ = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} (-1)^{m\mu'+n\nu'} \epsilon^{\frac{i\pi}{4}\phi(2m+\mu, 2n+\nu)} \\ \times \epsilon^{i(2m+\mu)x+i(2n+\nu)y} \end{cases}$$

where ϕ is the quadratic form (a, b, c); $i\pi a$, $i\pi b$, $i\pi c$ being the hyperbolic logarithms of A, B, C; and where the condition of convergence is that the real part of $i\phi$ must be a negative form of negative determinant.

Considering four different Theta functions, and writing

$$(13)\ \ldots\ldots \begin{cases} 2s = x_1+x_2+x_3+x_4, & 2t = y_1+y_2+y_3+y_4, \\ 2\sigma = \mu_1+\mu_2+\mu_3+\mu_4, & 2\tau = \nu_1+\nu_2+\nu_3+\nu_4, \\ 2\sigma' = \mu_1'+\mu_2'+\mu_3'+\mu_4', & 2\tau' = \nu_1'+\nu_2'+\nu_3'+\nu_4', \end{cases}$$

we have the formula

$$(14)\ldots \begin{cases} 4\Pi.\theta(\mu_j, \mu_j';\ \nu_j, \nu_j';\ x_j, y_j) \\ = \Sigma(-1)^{\alpha\sigma'+\alpha\alpha'+\beta\tau'-\beta\beta'} \\ \times \Pi.\theta(\sigma-\mu_j+\alpha,\ \sigma'-\mu_j'+\alpha';\ \tau-\nu_j+\beta,\ \tau'-\nu_j'+\beta';\ s-x_j,\ t-y_j) \end{cases}$$

the signs of multiplication Π referring to the four values 1, 2, 3, 4 of the index j, and the sign of summation Σ in the right-hand member referring to the symbols $\alpha, \alpha', \beta, \beta'$, each of which is to have the values 0, 1; so that the right-hand member contains sixteen terms. The formula may be demonstrated in the same manner as the corresponding formula for the single Theta functions (see the note already cited, "Proceedings," Vol. I., Part viii., Art. 2).

If, instead of a double Theta function, we consider a multiple Theta function containing λ arguments $x, y, \ldots$, and depending on λ pairs of indices $\mu\mu', \nu\nu', \rho\rho', \ldots$, we have the equation of definition

$$(15)\ \ldots\ldots \begin{cases} \theta(|\mu, \mu'|;\ |x|) = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \sum_{r=-\infty}^{r=+\infty} \ldots \\ (-1)^{m\mu'+n\nu'+r\rho'+\ldots} \times \epsilon^{\frac{i\pi}{4}\phi(2m+\mu,\ 2n+\nu,\ 2r+\rho,\ \ldots)} \\ \times \epsilon^{i(2m+\mu)x+i(2n+\nu)y+i(2r+\rho)z+\ldots} \end{cases}$$

where ϕ is a quadratic form, containing λ indeterminates, and such that the real part of $i\phi$ is definite and negative; and we obtain a formula

similar to (14); viz., taking four Theta functions, such that the sums of the homologous indices are all even, we have

$$(16)\ \ldots\ldots \begin{cases} 2^{\lambda}\,\Pi\,.\,\theta\,(\,|\,\mu_j,\ \mu'_j\,|\,;\ |\,x_j\,|\,) = \\ \Sigma\,(-1)^{|\,\alpha\sigma'+\alpha\alpha'\,|}\times\Pi\,.\,\theta\,(\,|\,\sigma-\mu_j+\alpha,\ \sigma'-\mu'_j+\alpha'\,|\,;\ |\,s-x_j\,|\,), \end{cases}$$

where the symbols $|\,\mu_j,\ \mu'_j\,|$, $|\,\sigma-\mu_j+\alpha,\ \sigma'-\mu'_j+\alpha'\,|$ are placed by abbreviation for the λ pairs

$$\mu_j,\ \mu'_j;\quad \nu_j,\ \nu'_j;\quad \ldots$$

and $$\sigma-\mu_j+\alpha,\ \sigma'-\mu'_j+\alpha';\quad \tau-\nu_j+\beta,\ \tau'-\nu'_j+\beta';\quad \ldots,$$

respectively; and the symbol $|\,\alpha\sigma'+\alpha\alpha'\,|$ represents the sum of the λ terms $$(\alpha\sigma'+\alpha\alpha')+(\beta\tau'+\beta\beta')+\ldots$$

Each of the 2λ indices $\alpha\alpha'$, $\beta\beta'$, ... is to receive successively the values 0, 1; and the sign of summation Σ extends to every combination of these values, so that the right-hand member consists of $2^{2\lambda}$ products, each affected with its proper sign, of four Theta functions.

The multiple Theta function (15) satisfies the equations

$$(17)\ \ldots\ldots \begin{cases} \theta\,(\mu+2,\ \mu';\ \ldots;\ x\ldots) = (-1)^{\mu'}\,\theta\,(\mu,\ \mu';\ \ldots;\ x\ldots), \\ \theta\,(\mu,\ \mu'+2;\ \ldots;\ x\ldots) = \theta\,(\mu,\ \mu';\ \ldots;\ x\ldots); \end{cases}$$

$$(18)\ \ldots\ldots\ \theta\,(\mu,\ \mu';\ \ldots;\ -x\ldots) = (-1)^{\mu\mu'}\,\theta\,(\mu,\ \mu';\ \ldots;\ x\ldots);$$

$$(19)\ \ldots\ldots \begin{cases} \theta\,(\mu,\ \mu';\ \nu,\ \nu';\ \rho,\ \rho';\ \ldots x,\ y,\ z\ldots) \\ \qquad = \epsilon^{\frac{i\pi}{4}\,\phi\,(\mu,\ \nu,\ \rho\ \ldots)}\times\epsilon^{i\,(\mu x+\nu y+\rho z+\ldots)} \\ \qquad\qquad \times\theta\,(0,\ 0;\ 0,\ 0;\ 0,\ 0;\ \ldots x_1,\ y_1,\ z_1,\ \ldots), \\ x_1 = x+\mu'\dfrac{\pi}{2}+\dfrac{\pi}{4}\dfrac{d\phi}{d\mu},\quad y_1 = y+\nu'\dfrac{\pi}{2}+\dfrac{\pi}{4}\dfrac{d\phi}{d\nu}, \\ z_1 = z+\rho'\dfrac{\pi}{2}+\dfrac{\pi}{4}\dfrac{d\phi}{d\rho},\quad \ldots\ldots, \end{cases}$$

which correspond to the equations (4), (10), (5), relating to a Theta function of one argument.

The equations (17) show that we need only attribute to the indices the values 0, 1; thus, the formula (16) may be regarded as comprehending 11^{λ} different equations, if we regard two equations as identical which may be deduced from one another by permutation of the four indices j; or, as comprehending $\dfrac{11\,.\,12\,.\,13\,\ldots\,10+\lambda}{1\,.\,2\,.\,3\,\ldots\,\lambda}$ different equations, if we also regard as identical two equations which may be deduced from one another by a simultaneous permutation of the λ arguments, and corresponding pairs of indices, in each of the Theta functions; viz., counting the different equations on this principle, we have as

many of them as there are λ-combinations, with repetition, of the eleven matrices of Table I. But we can always pass from any one of the equations (16) to any other by means of the formula (19), which may be employed to express any one of the 4^λ Theta functions as a product of any other by an exponential factor; although (for brevity) we have supposed that the indices of one of the Theta functions compared in that formula are all equal to zero.

Quaternion Proof of Minding's Theorem. *By* J. J. Walker, M.A.

[*Read February 13th*, 1879.]

At the conclusion of his very able paper on the Conditions of Astatic Equilibrium (Proc. of Lond. Math. Soc., Vol. IX., pp. 116—118), Professor Minchin has given a demonstration of this theorem,* in which recourse has been had to Cartesian coordinates, whereby the unity of treatment of the subject by Quaternions, elsewhere adopted throughout the paper, and to the use of which it owes much of its interest, has been marred. A short note, showing how the pure Quaternion proof may be completed, may, perhaps, be thought worthy of insertion.

Starting with Professor Minchin's equation (32) of the line in which the single resultant and Poinsot's axis combine, viz.,

$$\rho = \lambda i + \frac{1}{a} V(Jk - Kj),$$

with (31) $$SJk = SKj \quad \ldots\ldots (31),$$

which give $$\rho = \lambda i + \frac{1}{a}(Jk - Kj) \quad \ldots\ldots (32'),$$

wherein λ is an arbitrary scalar, the problem is to find the vector of the point in which that line meets the plane of *IJ* or *IK*.

Let $i' = UI$, $j' = UJ$, $k' = UK$, while $t_2 = TJ$, $t_3 = TK$.

To determine the λ of ρ in the plane *IJ*,

$$S\rho k' = \lambda Sik' + \frac{t_2}{a} Sj'kk',$$

i.e., $$0 = \lambda Sik' - \frac{t_2}{a} Ski',$$

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* Originally given in a paper which appeared in *Crelle*, Bd. XV. (1835), to which notice appears to have been more recently drawn by Moigno, *Mec. Anal.*, Ch. ix. (1868).

whence $$\lambda = \frac{t_2}{a}\frac{Sk\iota''}{Sik'} \quad \text{.....(34).}$$

Assume $$\rho = x\iota'' + yj'',$$

then, taking the scalars of $\rho\iota''$, $\rho j''$ successively, there result

$$x = -S\rho\iota'', \quad y = -S\rho j''.$$

Introducing in (32′) the value found for λ (34),

$$\rho aSik' = it_2 Sk\iota'' + Sik'\,(t_2 j''k - t_3 k'j),$$

whence $$S\rho\iota'' aSik' = Si\iota'' t_2 Sk\iota'' + Sik'\,(t_2 Skk' + t_3 Sjj''),$$

and $$S\rho j'' aSik' = Sij'' t_2 Sk\iota'' + Sik'\,(-t_3 Sj\iota''),$$

supposing j'', k', ι'' to be so designated that $j''k' = \iota''$...

Substituting in $$\rho = -\iota'' S\rho\iota'' - j'' S\rho j'',$$

$$\begin{aligned} -\rho aSik' &= \iota''\{t_2(Si\iota'' Sk\iota'' + Sik' Skk') + t_3 Sik' Sjj''\} \\ &\quad + j''(t_2 Sij'' Sk\iota'' - t_3 Sik' Sj\iota'') \\ &= i''(-t_2 Sij'' Skj'' + t_3 Sik' Sjj'') + j''(\ldots) \\ \text{by (31),} \quad &= \iota'' t_3(-Sij'' Sjk' + Sik' Sjj'') + j' t_3 \frac{Sij'' Sjk' Sk\iota'' - Sik' Sj\iota'' Skj''}{Skj''} \\ &= -\iota'' t_3 S\iota''k - j'' \frac{t_3}{Skj''}\{(Skj'')^2 - (Sjk')^2\}, \end{aligned}$$

i.e., $$\rho aSik' = \iota'' t_3 Sk\iota'' + j'' \sqrt{t_3^2 - t_2^2}\,\sqrt{(Skj'')^2 - (Sjk')^2},$$

—in virtue of $t_2 Skj'' = t_3 Sjk'$ (31)—which is the equation to a central conic, having $\frac{t_3}{a}\iota''$ and $\frac{\sqrt{t_3^2 - t_2^2}}{a} j''$ as axes, since

$$(Sik')^2 = (Sk\iota'')^2 + (Skj'')^2 - (Sjk')^2.$$

In fact $(Sik')^2 + (Sjk')^2 = \frac{1}{2} - \frac{1}{2} Skk'kk' = (Vkk')^2 = (S\iota'k)^2 + (Sj''k)^2$.

I may observe, also, that the quaternion equation of Poinsot's axis has been very simply worked out by pure quaternion process from its definition, without the assumption of any property deduced by other methods, by Prof. Tait, *Quaternions*, 2nd ed., Ch. xi.

[The Secretaries have received the following further Note on the same subject from Prof. Tait, which is printed here by order of the Council.]

Quaternion Proof of Minding's Theorem. *By* Prof. P. G. Tait.

1. Since the set of forces β_1 at α_1, &c., is to be turned about, let them be treated in connection with a rectangular system ι'', j'', k', *which is to turn with them.* We thus avoid the use of $q(\,)q^{-1}$, a weapon

obviously of immense power, the full art of managing which is unfortunately still unknown.

Then, taking the portions parallel to i' alone, they give

$$-\Sigma i' S i' \beta = -i' S i' \overline{\beta}, \text{ say,}$$

and a couple $-\Sigma V i' \alpha S i' \beta$, in the same plane.

They are therefore equivalent to a single force at the point

$$\frac{\Sigma \alpha S \beta i'}{S i' \overline{\beta}} = \frac{\phi i'}{S i' \overline{\beta}}, \text{ suppose, [centre of parallel forces].}$$

The three such points obviously lie in the plane $S\overline{\beta}\phi^{-1}\rho = 1$, which passes through the point $\phi\overline{\beta}^{-1}$. This, which we take as origin, is called the centre of the plane of centres, and will be the centre of the k' parallel forces if $\overline{\beta} = bk'$. In that case the other forces vanish, and their points of application go off to an infinite distance.

2. Hence (changing the α's in ϕ to suit the new origin) we may state the system as equivalent to the force bk' at the origin, and the couple $-V i' \phi i' - V j' \phi j'$. Since, by our choice of origin, we have made $\phi k' = 0$, $\phi i'$, $\phi j'$ are fixed vectors in the plane of centres, and we may take them as perpendicular to one another; for the condition is

$$S i' \phi' \phi j' = 0,$$

and $\phi'\phi$ is essentially a self-conjugate function. They are the central lines of the plane of centres. Let them be called $e_1 i$ and $e_2 j$, and let $\varpi\sigma = -e_1 i S i\sigma - e_2 j S j\sigma$. Thus we have now the force bk' at the origin, and the couple $V\varpi i i' + V\varpi j j'$.

The condition that these shall be equivalent to a single force is, of course,

$$Sk' (\varpi i i' + \varpi j j') = 0.$$

This enables us to express i' and j' in terms of k', and thus to eliminate them. For we have at once

$$t i' = \quad Vk'\varpi j + k' Vk' \varpi i,$$

$$t j' = -Vk'\varpi i + k' Vk' \varpi j,$$

where t is the common tensor, a function of k'. One simple form is

$$t = S i' \varpi i + S j' \varpi j.$$

Now the equation of the line in which the single force acts is

$$bVk'\rho = V(\varpi i i' + \varpi j j'),$$

or, with the above values of i', j',

$$-bt\rho = (\varpi^2 + u) k' - e_1 e_2 k,$$

where u is wholly indeterminate.

But Minding's theorem refers to the intersection of this line with the plane

$$Si\rho = 0,$$

or

$$Sj\rho = 0.$$

Take the first, and we get

$$(e_1^2 + u)\, Sik' = 0,$$

which determines u, and then at once

$$-bt\,(\varpi^2 - e_1^2)^{-1}\rho = k' + \frac{e_2}{e_1}k.$$

Now we see easily that

$$t^2 = S\left(k' + \frac{e_2}{e_1}k\right)\left[(\varpi^2 - e_1^2)\,k' - e_1 e_2 k\right];$$

so that, finally,

$$S\rho\,(\varpi^2 - e_1^2)^{-1}\rho = \frac{1}{b^2},$$

which is one of the focal conics of the system

$$S\rho\,(\varpi^2 - h)^{-1}\rho = \frac{1}{b^2}.$$

Minding's theorem might have been anticipated from the general equation above; for it gives at once the transformation

$$b^2 S\rho\,(\varpi^2 + u)^{-1}\rho = 1 - \frac{1}{t^2 u}(e_1^2 + u)(e_2^2 + u),$$

which is obviously much simplified when $u = -e_1^2$ or $= -e_2^2$. This is a good example of the suggestiveness of quaternions.

March 13th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. J. D. H. Dickson, M.A., was admitted into the Society. Mr. R. Hargreaves and Prof. W. E. Story were elected Members. Mr. Donald McAlister, B.A., B.Sc., Fellow of St. John's College, Cambridge, was proposed for election.

Prof. Cayley made a few remarks appreciative of the late Prof. Clifford's work as a mathematician, instancing more particularly his papers on the Riemann's Surface, the Contact Problems of Mr. Spottiswoode, and the Classification of Loci.

The Chairman, the Rev. A. Freeman, and Dr. Hirst, also added a few words on the loss the Society and the mathematical world generally

had sustained,* and expressed a hope that steps would be taken to secure the publication, if necessary, of any MSS. Prof. Clifford might have left.

Dr. Hirst made a statement to the Society of the present position of the movement for presenting a De Morgan memorial Medal to the Society; it appeared that the bust and die for the Medal had been executed by Mr. Woolner, and that, after all claims for these had been met, there would still be a small sum required to make up the total requisite for investment in the Funds to bring in the necessary interest for the above purpose. It was agreed that a subscription list be opened for any Members who might wish to contribute to the Fund, and that Mr. Tucker be authorized to receive such subscriptions as might be paid in in this way, and to hand them over to Mr. A. Wills, Q.C., the Hon. Sec. to the Fund. Copies of the obverse and reverse of the Medal were exhibited.

The following communications were made:—

"On Differential Equations, Total and Partial; and on a new Soluble Class of the First, and an Exceptional Case of the Second;" by Sir James Cockle, F.R.S.

"Discussion of two Double Series arising from the number of Terms in Determinants of certain Forms;" by Mr. J. D. H. Dickson.

"Two Geometrical Notes relating to Surfaces of the Second Order;" by Prof. H. J. S. Smith, F.R.S.

The following presents were received:—

"Educational Times," March, 1879.

"Atti della R. Accademia dei Lincei . . . Transunti," Vol. iii., Fasc. 3°, Febb. 1879; Roma 1879.

"Bulletin de la Société Mathématique de France," Tome vii., No. 2; Paris, 1879.

"The Physical Society of London, Proceedings," Vol. ii., Pt. v., July to December, 1878; London, 1879.

"Proceedings of Royal Society," Vol. xxviii., No. 193.

"Crelle's Journal," Bd. 86, viertes Heft; Bd. 87, erstes Heft und zweites Heft.

"Monatsbericht," December, 1878; Berlin, 1879.

"New Solar Elements;" by Rev. G. T. Carruthers; Nagpur, 1879: from the Author.

"Proceedings of Royal Society of Edinburgh," Session 1877–78, Vol. ix., No. 100, Nov. 1877 to July 1878.

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* Prof. Clifford died at Madeira, March 3rd, 1879.

On Differential Equations, Total and Partial; and on a New Soluble Class of the First and an Exceptional Case of the Second. By Sir James Cockle, F.R.S.

[*Read March 13th*, 1879.]

1. The contents of the seven sections of this paper are substantially as follows:—§1 and §2 are preliminary; §2, consisting chiefly of matter intended to be illustrative, embraces Binary forms (such as $Pdx+Qdy$), with some reference to Ternaries (such as $Pdx+Qdy+Rdz$); §3 is on a First, and well-marked, Species of Exceptional Single Solution of Ternaries; §4 is a Digression on Exact Ternary Differentials, and on the process of Monge; §5 is on Exceptional Single Solutions of Ternaries, considered generally; §6 is on the same subject, considered in relation to partial differential equations; §7 is on Quaternaries. There are some points into which I should like to have entered more fully, but the paper has already run to some length.

§ 1. *Notation.—Lemmas.*

2. Unless the contrary appear, the following notation will be preserved throughout this paper. The equations, wherein p and q are in general functions of x, y, and z,

$$\Delta = \frac{d}{dx} + p\frac{d}{dz}, \quad \text{and} \quad \nabla = \frac{d}{dy} + q\frac{d}{dz}$$

define the operators Δ and ∇, while $\square$ standing alone is given by

$$\square = \nabla p - \Delta q,$$

which, developed, gives

$$\square = \frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz}.$$

3. From the compounded operator $\nabla\Delta - \Delta\nabla$ (and the same thing would be true of an operator similarly compounded of any two operators each of the form $\Sigma p_n \frac{d}{dx_n}$) all will disappear which would disappear if $\frac{d}{dx}$, $\frac{d}{dy}$, and $\frac{d}{dz}$ were symbols of quantity; *ex. gr.*, all such symbols as $\frac{d^2}{d\xi d\eta}$, and that whether ξ and η be equal or unequal. And

$$\nabla\Delta - \Delta\nabla = \square\frac{d}{dz}.$$

4. When $\square$ stands not alone, but in this combination, viz., $\square_2(P, Q, R)$, then

$$\square_2(P, Q, R) = P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right),$$

so that the solitary $\square$ means, in fact,

$$-\square_2(p, q, -1).$$

5. Since

$$P\theta\left(Q\frac{d\theta}{dz} - R\frac{d\theta}{dy}\right) + Q\theta\left(R\frac{d\theta}{dx} - P\frac{d\theta}{dz}\right) + R\theta\left(P\frac{d\theta}{dy} - Q\frac{d\theta}{dx}\right)$$

vanishes identically, and since always

$$A\frac{d\theta}{da} = \frac{d}{da}(A\theta) - \theta\frac{dA}{da},$$

consequently $\square_2(P\theta, Q\theta, R\theta) - \theta^2\square_2(P, Q, R)$

vanishes identically; *i.e.*, we have identically

$$\square_2(P\theta, Q\theta, R\theta) = \theta^2\square_2(P, Q, R).$$

6. The symbols A, E, and O will be taken to refer to x, y, and z respectively; A denoting a function of x only, E a function of y only, and O a function of z only. In conformity with this notation, $\mathcal{E}$ will denote a function of x and y only, $\mathfrak{E}$ a function of y and z only, and we may take Θ to represent a function of x and z only.

§ 2. *On certain Properties of Binaries.—Reference to Ternaries.*

7. If the expressions

$$dz - (\Theta + p)\,dx \quad \text{and} \quad dz - p\,dx$$

have a common integrating factor μ, then μ satisfies

$$-\frac{d}{dz}(\mu\Theta + \mu p) = \frac{d\mu}{dx} = -\frac{d}{dz}(\mu p),$$

whence

$$0 = -\frac{d}{dz}(\mu\Theta), \quad \text{or} \quad \mu = \frac{1}{A\Theta};$$

and

$$\frac{d}{dx}\frac{1}{A\Theta} = -\frac{d}{dz}\frac{p}{A\Theta}$$

may be put under the form

$$\Delta\left(\frac{1}{A\Theta}\right) + \frac{1}{A\Theta}\frac{dp}{dz} = 0,$$

or, developing, reducing, and transposing,

$$\frac{dp}{dz} - \Delta \log \Theta = \frac{1}{A} \frac{dA}{dx},$$

and, if the common integrating factor exists, the sinister of the last equation will of course be a function of x only. We speak of a common integrating factor even when one of the expressions is already an exact differential.

8. If the two expressions

$$dz - (\Theta + p)\, dx \quad \text{and} \quad dz - p\, dx$$

have a common solution, *i.e.*, if there be a substitution which will reduce each to zero, then that solution is, or is contained in, $\Theta = 0$. Take, for example,

$$dz + 2\sqrt{-z}\,.\,dx \quad \text{and} \quad dz + 2x\, dx.$$

9. When two expressions have both a common integrating factor and a common solution, the solution makes the factor infinite. But no necessary inference can thence be drawn as to singularity. Thus, the common solution of

$$dz - (x + \sqrt{x^2 - 2z})\, dx \quad \text{and} \quad dz - x\, dx$$

is singular with respect to the first expression, and particular with respect to the second; while the common solution of

$$dz - \{1 + (z - x)\}\, dx \quad \text{and} \quad dz - dx$$

is a particular primitive of both.

10. Applying a portion of the foregoing to ternaries, the condition that the two expressions

$$dx + z\, dy + y\, dz \quad \text{and} \quad dx + z\, dy - \frac{x}{z}\, dz$$

should have a common solution is $\frac{x}{z} + y = 0$. And this is sufficient, for the first expression is $d(x + zy)$, and the second is $z d\left(\frac{x}{z} + y\right)$, the complete integrals are $x + zy = c$ and $\frac{x}{z} + y = c$, and the common solution is given by $c = 0$.

§ 3. *On a First, and well-marked, Species of Exceptional Single Solution of Ternaries.*

11. Let N, M and L be the integrating factors of $X dx + Y dy + Z dz$, when x, y, and z are respectively taken as constant. Then, to say that $\square_2$ vanishes identically, is in effect to say that all (both) the three (substantially two) equations $N = M = L$ can be satisfied.

But it is sometimes possible, even when $\square_2$ does not so vanish, to satisfy one of the equations, say $N = M$, and in such case M (or N) can be at once determined.

The result of such determination is to give to $M(Xdx + Ydy + Zdz)$ either of the forms, indifferently,

$$\frac{dV}{dz}dz + \frac{dV}{dx}dx + MYdy, \text{ or } \frac{dW}{dz}dz + \frac{dW}{dy}dy + MXdx,$$

which may be replaced by

$$dV - V_2 dy, \text{ or } dW - W_2 dx.$$

12. Subtracting one of these equivalent forms from the other, we have the identity $d(V-W) = V_2 dy - W_2 dx$,

and, the sinister being an exact differential, the dexter is also, and V_2 and W_2 are functions of x and y only, satisfying the identity

$$\frac{dV_2}{dx} + \frac{dW_2}{dy} = 0.$$

Hence, if we represent V_2 by $\frac{d\text{Æ}_2}{dy}$, and therefore W_2 by $-\frac{d\text{Æ}_2}{dx}$, the two forms will become $dV - \frac{d\text{Æ}_2}{dy}dy$, and $dW + \frac{d\text{Æ}_2}{dx}dx$.

13. To determine M (or N), we have

$$\frac{d(ZM)}{dx} = \frac{d(XM)}{dz} \text{ and } \frac{d(ZM)}{dy} = \frac{d(YM)}{dz},$$

or, writing for a few moments

$$M = \frac{\mu}{Z}, \quad \frac{X}{Z} = -p, \text{ and } \frac{Y}{Z} = -q,$$

$$\frac{d\mu}{dx} = -\frac{d(p\mu)}{dz}, \text{ and } \frac{d\mu}{dy} = -\frac{d(q\mu)}{dz}.$$

Hence $$\frac{d\mu}{dx} = -p\frac{d\mu}{dz} - \mu\frac{dp}{dz},$$

or $$\frac{dp}{dz} = \Delta \log \frac{1}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

so $$\frac{dq}{dz} = \nabla \log \frac{1}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Operating on (1) with ∇ and on (2) with Δ, we have, subtracting the results, $$\frac{d}{dz}(\nabla p - \Delta q) = (\nabla\Delta - \Delta\nabla)\log\frac{1}{\mu},$$

or
$$\frac{d\square}{dz} = \square \frac{d}{dz} \log \frac{1}{\mu},$$

whence
$$\frac{1}{\mu} = Æ\square,$$

where $Æ$ is a function of x and y only. This relation is necessary, but not sufficient, for the solution of (1) and (2). Substituting therein, we find
$$\frac{dp}{dz} - \Delta \log \square = \frac{1}{Æ} \frac{dÆ}{dx} \text{.....(i.)},$$
$$\frac{dq}{dz} - \nabla \log \square = \frac{1}{Æ} \frac{dÆ}{dy} \text{.....(ii.)};$$

or, replacing p and q by $-\frac{X}{Z}$ and $-\frac{Y}{Z}$ respectively, and remembering that
$$\square = -\square_2 \left(-\frac{X}{Z}, -\frac{Y}{Z}, -1\right)$$
$$= -\square_2 \left(-\frac{X}{Z}, -\frac{Y}{Z}, -\frac{Z}{Z}\right) = \frac{-\square_2(X, Y, Z)}{Z^2},$$

we have
$$-\frac{d}{dz}\left(\frac{X}{Z}\right) - \left(\frac{d}{dx} - \frac{X}{Z}\frac{d}{dz}\right) \log\left\{-\frac{\square_2(X, Y, Z)}{Z^2}\right\} = \frac{1}{Æ}\frac{dÆ}{dx} \text{...(i.)},$$
$$-\frac{d}{dz}\left(\frac{Y}{Z}\right) - \left(\frac{d}{dy} - \frac{Y}{Z}\frac{d}{dz}\right) \log\left\{-\frac{\square_2(X, Y, Z)}{Z^2}\right\} = \frac{1}{Æ}\frac{dÆ}{dy} \text{...(ii.)};$$

and, when the conditions (i.) and (ii.) are identically satisfied, they are sufficient, and $Xdx + Ydy + Zdz$ can be put, indifferently, under either of the forms
$$\frac{-Æ\square_2}{Z}\left(dV - \frac{dÆ_2}{dy}dy\right) \text{ or } \frac{-Æ\square_2}{Z}\left(dW + \frac{dÆ_2}{dx}dx\right),$$

where $\square_2$ is written for $\square_2(X, Y, Z)$.

14. Had we wished to satisfy $M = L$, we should have, in place of the last set of formulæ marked (i.) and (ii.), the following, viz.,
$$-\frac{d}{dx}\left(\frac{Y}{X}\right) - \left(\frac{d}{dy} - \frac{Y}{X}\frac{d}{dx}\right) \log\left(-\frac{\square_2}{X^2}\right) = \frac{1}{Œ}\frac{dŒ}{dy} \text{.....(iii.)},$$
$$-\frac{d}{dx}\left(\frac{Z}{X}\right) - \left(\frac{d}{dz} - \frac{Z}{X}\frac{d}{dx}\right) \log\left(-\frac{\square_2}{X^2}\right) = \frac{1}{Œ}\frac{dŒ}{dz} \text{.....(iv.)},$$

for $Xdx + Ydy + Zdz$ and $\square_2(X, Y, Z)$ are unaffected by the simultaneous cyclical interchanges (X, Y, Z), (x, y, z), and N, M, L correspond respectively with the cases of x, y, z constant. If, then, (i.) and (ii.) represent the phase (X, Y, Z) (x, y, z), the phase (Y, Z, X) (y, z, x)

will be represented by (iii.) and (iv.), and the phase (Z, X, Y) (z, x, y) by

$$-\frac{d}{dy}\left(\frac{Z}{Y}\right)-\left(\frac{d}{dz}-\frac{Z}{Y}\frac{d}{dy}\right)\log\left(-\frac{\square_2}{Y^2}\right)=\frac{1}{\Theta}\frac{d\Theta}{dz} \quad \text{.....(v.)},$$

$$-\frac{d}{dy}\left(\frac{X}{Y}\right)-\left(\frac{d}{dx}-\frac{X}{Y}\frac{d}{dy}\right)\log\left(-\frac{\square_2}{Y^2}\right)=\frac{1}{\Theta}\frac{d\Theta}{dx} \quad \text{.....(vi.)},$$

which system answers to $L = N$. The three equations $N=M$, $M=L$, and $L = N$ may indeed be appropriately symbolized by $(x) = (y)$, $(y) = (z)$, and $(z) = (x)$.

By an independent process I have arrived at the results

$$\frac{1}{\mu}=\frac{\Theta\square}{q}=\frac{(-\Theta)(-\square)}{q},$$

and

$$\frac{1}{q}\left(\frac{dp}{dy}-\frac{dq}{dx}\right)-\left(\frac{p}{q}\frac{d}{dy}-\frac{d}{dx}\right)\log\frac{-\square}{q}=-\frac{1}{\Theta}\frac{d\Theta}{dx},$$

$$\frac{1}{q}\left(\frac{dq}{dz}-\nabla\log\frac{\square}{q}\right)=\frac{1}{\Theta}\frac{\Theta}{dz},$$

wherein, however, $p=-\frac{X}{Z}$, and $q=-\frac{Y}{Z}$. On making these substitutions, and reducing, the latter results coincide with (v.) and (vi), and thus we have a verification; in connection with which I remark that $Æ$, $Œ$, and Θ may each be affected with an arbitrary constant multiplier; for $d\log(Cv)=d\log v$.

The transformation here indicated is that of $Xdx+Ydy+Zdz$ into

$$-\frac{\Theta\square_2}{Y}\left(dU+\frac{d\Theta_2}{dz}dz\right) \text{ or } -\frac{\Theta\square_2}{Y}\left(dW-\frac{d\Theta_2}{dx}dx\right).$$

16. So for $M=L$ the transformation indicated is that of $Xdx+Ydy+Zdz$ into

$$-\frac{Œ\square_2}{X}\left(dU-\frac{dŒ_2}{dz}dz\right) \text{ or } -\frac{Œ\square_2}{X}\left(dV+\frac{dŒ_2}{dy}dy\right).$$

17. Recurring to the first form of (i.) and (ii.), and multiplying (i.) into dx and (ii.) into dy, add the products, and add and subtract $\frac{d\square}{\square dz}dz$ to and from the result. We shall obtain

$$\frac{dp}{dz}dx+\frac{dq}{dz}dy-\frac{d\square}{\square}+\frac{1}{\square}\frac{d\square}{dz}(dz-pdx-qdy)=\frac{1}{Æ}dÆ \quad \text{.....(A)},$$

which gives, on integration, $\log Æ$. If we wish to find $d\log(Æ\square)$

at once, and write for a moment $\frac{1}{\square}\frac{d\square}{dz} = \frac{1}{\rho}$, then we have

$$\frac{1}{\rho}\left\{dz + \left(\rho\frac{dp}{dz} - p\right)dx + \left(\rho\frac{dq}{dz} - q\right)dy\right\} = d\log(\text{Æ}\square) \quad \ldots(\text{B}),$$

and, the dexter of (B) being an exact differential, the sinister is so, and integration will give $\log(\text{Æ}\square)$. And (B) is very readily applicable to $dz - pdx - qdy$ when p and q (or p or q) are linear in z.

18. Cases in which, when one of the systems (i.) and (ii.), or (iii.) and (iv.), or (v.) and (vi.), is identically satisfied, $\square = 0$ solves $dz - pdx - qdy$, exist, and some are given in this paper. The mark of such a solution is, that it accompanies one of the systems, and at the same time discloses another solution (or rather two other solutions) of Monge's form. This species of solution seems to be suggested when we suppose one, and only one, of the relations $N = M = L$ to be capable of being fulfilled, and so to lie in the pathway of research.

19. It will, I think, be found convenient to give a name to the functions $\square$ and $\square_2$. Let us call them Discriminoids. I find that the solution of the following example, taken from a paper "On Ternary Differential Equations," by Mr. Robert Rawson, ("Manchester Proceedings," Vol. XVI., p. 114), and wherein w, v are functions of x, y only (w, however, not being of the form $fxFy$); viz., the solution of

$$\left(\frac{dw}{dy}z - \frac{dv}{dy}\right)dy + \left(\frac{v}{w}\frac{dw}{dx} - \frac{dv}{dx}\right)dx + wdz,$$

belongs to this First Species of Discriminoidal Solution. I shall give another such example further on.

§4. *Digression on Exact Ternary Differentials and on the Process of Monge.*

20. The necessity that the three conditions

$$\frac{dP}{dy} = \frac{dQ}{dx}, \quad \frac{dQ}{dz} = \frac{dR}{dy}, \text{ and } \frac{dR}{dx} = \frac{dP}{dz}$$

should be fulfilled in order that $Pdx + Qdy + Rdr$ should be an exact differential, is at once recognized. The sufficiency of these conditions may be easily proved. They, in effect, demand that $Pdx + Qdy + Rdz$ should be capable of being placed under any of the three equivalent forms $dU + \left(R - \frac{dU}{dz}\right)dz$; $dV + \left(Q - \frac{dV}{dy}\right)dy$; $dW + \left(P - \frac{dW}{dx}\right)dx$ indifferently. Subtracting the second from the first, the third from

the second and the first from the third successively, we obtain, after transposition, the following identities :—

$$d(U-V) = \left(Q-\frac{dV}{dy}\right)dy-\left(R-\frac{dU}{dz}\right)dz,$$

$$d(V-W) = \left(P-\frac{dW}{dx}\right)dx-\left(Q-\frac{dV}{dy}\right)dy,$$

$$d(W-U) = \left(R-\frac{dU}{dz}\right)dz-\left(P-\frac{dW}{dx}\right)dx.$$

But, the sinisters being exact differentials, the dexters are so. Consequently

$Q-\frac{dV}{dy}$ and $R-\frac{dU}{dz}$ are free from x,

$P-\frac{dW}{dx}$ and $Q-\frac{dV}{dy}$ are free from z,

$R-\frac{dU}{dz}$ and $P-\frac{dW}{dx}$ are free from y;

in other words, $R-\frac{dU}{dz}$ is a function of z only, $Q-\frac{dV}{dy}$ is a function of y only, and $P-\frac{dW}{dx}$ a function of x only. This establishes the sufficiency.

21. I question whether Monge's process can be advantageously applied to the finding of single solutions involving all the variables. For, first, in applying it we may lose a solution; secondly, there is a better procedure; and, thirdly, it may be doubted whether his solution is, in a proper sense, general. If by any means we can reduce $Pdx+Qdy+Rdz$ to the form $dV-\rho dW$; then $V=\phi(W)$, combined with $\rho=\phi'(W)$, would be a dual solution having apparently as strong a claim to generality as that of Monge. Using Mr. Harley's cyclical symbol Σ', form $\Sigma' zdx$ or $zdx+xdy+ydz$; then, using the integrating factor $\frac{1}{zx}$ for the first two terms, we reduce $\Sigma' zdx$ to

$$d\left(\log x+\frac{y}{z}\right)+\frac{y}{z}\left(\frac{1}{z}+\frac{1}{x}\right)dz;$$

whence the solution of Monge, viz.,

$$\log x+\frac{y}{z}=\phi(z), \quad -\frac{y}{z}\left(\frac{1}{z}+\frac{1}{x}\right)=\phi'(z).$$

But by cyclical changes we obtain, through Monge's method, two more

solutions each with a claim, as strong as that of the foregoing, to be called general, viz.,

$$\log y + \frac{z}{x} = \phi(x), \quad -\frac{z}{x}\left(\frac{1}{x}+\frac{1}{y}\right) = \phi'(x),$$

and

$$\log z + \frac{x}{y} = \phi(y), \quad -\frac{x}{y}\left(\frac{1}{y}+\frac{1}{z}\right) = \phi'(y).$$

Nor is this all. The aggregate of these three solutions does not appear to constitute a general solution. Assume

$$ax^2+by^2+cz^2+2dxy+2exz+2fyz = V, \quad -\rho = 2ax+2\beta y+2\gamma z,$$

and

$$W = Ax+By+Cz,$$

wherein all the coefficients are constant, and put

$$dV-\rho dW = 2\Sigma' z\,dx = 2(z dx+x dy+y dz);$$

then we shall find that the six quantities

$$a+A\alpha,\ d+A\beta,\ b+B\beta,\ f+B\gamma,\ c+C\gamma, \text{ and } e+C\alpha$$

must each be equated to zero, and the three

$$e+A\gamma,\ d+B\alpha, \text{ and } f+C\beta$$

each to unity. Hence we shall be led to

$$\alpha(B+C) = A(\beta+\gamma), \quad \gamma(A+B) = C(\alpha+\beta),$$

and

$$\beta(C+A) = B(\alpha+\gamma);$$

and if we make $B+C=0$, $A=0$, $\alpha+\gamma=0$, and $\beta=0$, we find $a, d, b, =0$; $e, f, =1$; $B\alpha=1$; $B\gamma=C\alpha=-1$; and $c=-C\gamma$. Now put $B=1$, $C=-1$, and we have $\alpha=1$ and $\gamma=-1$. Hence $V=-z^2+2xz+2yz$; $-\rho=2x-2z$; $W=y-z$; consequently

$$dV = 2\{(x+y-z)dz+z dx+z dy\},$$

$$-\rho dW = 2\{-(x-z)dz+x dy-z dy\};$$

and therefore

$$dV-\rho dW = 2(z dx+x dy+y dz).$$

Thus, for $z dx+x dy+y dz=0$, we obtain the dual solution

$$2xz+2yz-z^2 = \phi(y-z); \quad -2(x-z) = \phi'(y-z);$$

and hence, by cyclical changes, the two additional dual solutions

$$2yx+2zx-x^2 = \phi(z-x); \quad -2(y-x) = \phi'(z-x);$$

and

$$2zy+2xy-y^2 = \phi(x-y); \quad -2(z-y) = \phi'(x-y).$$

22. In these last three solutions the function ϕ is arbitrary, and if we take ϕ as an algebraical function, we may represent $z dx+x dy+y dz=0$ by curves resulting from the intersection of surfaces represented by algebraical equations. No logarithm need enter.

23. By assuming $Pdx+Qdy+Rdz = dV-\rho\, dW$, and seeking for particular solutions of the partial differential equations thence deducible, we may possibly sometimes obtain solutions not within Monge's formula. In the concluding solution of the last example, there was no need for any process whatever of integration, all the steps being purely algebraical.

5. *On Exceptional Single Solutions of Ternaries, considered generally.*

24. Changing the notation for the purposes of this section, I shall now endeavour to ascertain the conditions, necessary and sufficient, in order that $dz-Mdx-Ndy$ may, when not completely integrable, admit of a single solution involving all the variables.

25. Assume that there is such a solution, represent it by $u = 0$, and let $dz-pdx-qdy = 0$ be its (divided if necessary) differential equation. Then, if the two expressions $dz-Mdx-Ndy$ and $dz-pdx-qdy$ have a common solution, that solution gives $M = p$ and $N = q$, and is, or is contained in, either, or both, of the latter equations, one of which, however, may be an identity (*ex. gr.*, $M = p$ identically).

26. Of the two forms $\square$ and $\square_2$, which are connected by the relation $\square\,(P, Q, R) = -\square_2\,(P, Q, R)$, I shall here use $\square$ and form it as before, save that p and q are to be replaced by M and N respectively.

Thus
$$\square = \frac{dM}{dy}+N\frac{dM}{dz}-\frac{dN}{dx}-M\frac{dN}{dz}.$$

27. Now put $M = P+p$ and $N = Q+q$, and develope. We have

$$\begin{aligned}\square = \quad & \frac{dP}{dy}+Q\frac{dP}{dz}-\frac{dQ}{dx}-P\frac{dQ}{dz}\\ & +q\frac{dP}{dz}+Q\frac{dp}{dz}-p\frac{dQ}{dz}-P\frac{dq}{dz}\\ & +\frac{dp}{dy}-\frac{dq}{dx}+q\frac{dp}{dz}-p\frac{dq}{dz}.\end{aligned}$$

28. But, in this section, $dz-pdx-qdy = 0$ is supposed to be the differential equation of a given primitive $u = 0$. Hence the last line of Art. 27 vanishes identically. Expunging it, and rearranging the remaining terms, we have

$$\begin{aligned}\square = \quad & \frac{dP}{dy}+q\frac{dP}{dz}-\frac{dQ}{dx}-p\frac{dQ}{dz}\\ & +Q\frac{d}{dz}(P+p)-P\frac{d}{dz}(Q+q).\end{aligned}$$

29. Since p is, in this section, the equivalent of $\frac{dz}{dx}$, and q of $\frac{dz}{dy}$, the first two terms of $\square$ in Art. 28 may be written $\frac{dP}{dy}+\frac{dP}{dz}\frac{dz}{dy}$. And this last expression vanishes. For $u=0$ gives $M=p$, and $M=p$ gives $P=0$, and z is a function of the independent variables x and y. Consequently, the differential coefficient of P with respect to y, treating z in P as a function of x and y, vanishes. For like reasons the aggregate of the next two terms, viz., $-\left(\frac{dQ}{dx}+p\frac{dQ}{dz}\right)$, vanishes.

30. Thus, $u=0$ reduces $\square$ to

$$\square = Q\frac{d}{dz}(P+p)-P\frac{d}{dz}(Q+q),$$

or to

$$\square = Q\frac{dM}{dz}-P\frac{dN}{dz},$$

and $u=0$ [which indeed might be replaced by $z=f(x, y)$] is accompanied by $P=0$ and $Q=0$.

31. When P and Q both vanish, either

$$\square = 0,$$

or

$$\frac{dM}{dz}=\infty,$$

or

$$\frac{dN}{dz}=\infty.$$

32. Whether any equation herein contained is or is not a solution of the proposed differential equation $(dz-Mdx-Ndy=0)$, must be determined by trial.

33. If no single solution involving *all* the variables be thus obtained, then there exists no such solution, unless, indeed, the making x or y, instead of z, the dependent variable, should lead to one.

34. The precise bearing of the foregoing on the subject of singular solution I have not investigated. But the following example will illustrate all, or most, of what precedes. I do not know that I have exhausted the illustrations. M and N having served their purpose, we may now replace them by p and q, giving, henceforth, to p and q their old and unrestricted meanings, and no longer supposing that $dz-pdx-qdy$ is completely integrable by a factor. And $\frac{dp}{dz}=\infty$ and $\frac{dq}{dz}=\infty$ replace the conditions $\frac{dM}{dz}=\infty$ and $\frac{dN}{dz}=\infty$.

35. Take the expression

$$dz-(\alpha u^m+\beta u-1)\,dx-(\gamma u^m+\delta u^n+\epsilon u-1)\,dy,$$

wherein $$u=x+y+z,$$

and $$\square=(m-n)\,\alpha\delta u^{m+n-1}+(m-1)\,(\alpha\epsilon-\beta\gamma)\,u^m-(n-1)\,\beta\delta u^n.$$

36. First let m and n be positive. Then

(I.) If $m=n=1$, the expression is integrable by a factor, for $\square$ vanishes identically.

(II.) If $m=n$, then

$$\square=(m-1)\,(\alpha\epsilon-\beta\gamma-\beta\delta)\,u^m,$$

and $\square=0$ gives the solution $u=0$. The expression and its solution are of the first species (discussed in § 3) of discriminoidal solution, and

$$\mathcal{E}=e^{(1-m)(\beta x+\epsilon y)}.$$

(III.) If $n=1$, then

$$\square=(m-1)(\alpha\delta+\alpha\epsilon-\beta\gamma)\,u^m,$$

and $\square=0$ gives the solution $u=0$. The expression and its solution are of the same first species, and

$$\mathcal{E}=e^{(1-m)[\beta x+(\delta+\epsilon)y]}.$$

(IV.) If $m=1$, then

$$\square=(1-n)\,(\alpha\delta+\beta\delta)\,u^n,$$

and $\square=0$ gives the solution $u=0$. The species is the same, and

$$\mathcal{E}=e^{(1-n)[(\alpha+\beta)x+(\gamma+\epsilon)y]}.$$

(V.) If m and n have other values or relations, and if $m+n$ be >1, then $\square=0$ gives the solution $u=0$, which now belongs to a second species of discriminoidal solution.

(VI.) If $m+n$ be <1, then, since $\square=\infty$ accompanies the solution $u=0$, that solution may be said to belong to a second genus of discriminoidal solution. This solution is of course within the purview of Art. 31.

(VII.) If m and n be unequal, and $m+n=1$, then

$$\square=(2m-1)\,\alpha\delta+(m-1)\,(\alpha\epsilon-\beta\gamma)\,u^m+m\beta\delta u^{1-m},$$

wherein m is not $\frac{1}{2}$, and m and $1-m$ are positive. Here $u=0$ is an exceptional solution, not in any sense discriminoidal, for it does not satisfy either $\square=0$ or $\square=\infty$. It, in fact, makes $\square=(2m-1)\,\alpha\delta$, and comes from $\frac{dp}{dz}=\infty$ and $\frac{dq}{dz}=\infty$ indifferently. For these

equations are respectively equivalent to

$$-ma u^{m-1} - \beta = \infty \quad \text{and} \quad -m\gamma u^{m-1} - n\delta u^{n-1} - \epsilon = \infty,$$

each of which is satisfied by $u = 0$.

37. Let $\beta = 0$ and $\epsilon = 0$, and let both m and n be negative. Here $\square = 0$ yields $u = \infty$, a discriminoidal solution of the second species of the first genus of discriminoidal solution, for the system (i.) and (ii.) can now only be satisfied by $\alpha, \gamma, \delta, = 0$.

§ 6. *The same subject considered in relation to Partial Differential Equations.*

38. The two expressions

$$\frac{d\theta}{dz}(dz - pdx - qdy)$$

and

$$d\theta - \left(\frac{d\theta}{dx} + p\frac{d\theta}{dz}\right)dx - \left(\frac{d\theta}{dy} + q\frac{d\theta}{dz}\right)dy$$

are identical, and if we can make the latter vanish, the former will vanish also. The latter will be reduced to zero provided that we can find θ as a function of x, y, and z from the two partial differential equations

$$\frac{d\theta}{dx} + p\frac{d\theta}{dz} = 0 \quad \text{and} \quad \frac{d\theta}{dy} + q\frac{d\theta}{dz} = 0,$$

for then $\theta = 0$ (or $\theta = \text{const.}$) will make the second expression vanish. But if we follow the process given in Boole's "Differential Equations" (1865, Suppl., p. 88), we shall merely be led, from the partial differential equations, back to a total differential equation

$$dz - pdx - qdy = 0.$$

That is to say, we shall be so led if we confine our attention to solutions of one step, by which I mean solutions which satisfy a partial differential equation solely in virtue of the values which they give to the partial differential coefficients. Boole (*ibid.* 1865, pp. 331, 332; Suppl., pp. 60, 61) does not seem to recognize any distinction between solutions of one and of two steps. Yet for a solution of two steps his functional determinant would not vanish identically.

39. By way of illustration, take

$$(z^2 + xz - y^2 - xy)\frac{dz}{dx} + (x^2 + xy - z^2 - zy)\frac{dz}{dy} - (y^2 + yz - x^2 - xz) = 0 \quad \ldots\ldots\ldots(B),$$

and let $x + y + z = 0$. This is a solution of one step, for it gives $\frac{dz}{dx} = -1$ and $\frac{dz}{dy} = -1$, and these values reduce the sinister of (B) to an identical zero.

40. But let a be an unreal cube root of unity, and let $z+ax+a^2y=0$, which gives $\frac{dz}{dx}=-a$ and $\frac{dz}{dy}=-a^2$.

These values do not reduce the sinister to zero, but the equation itself is reduced to $(1-a)\,a\,(x+y+z)(z+ax+a^2y)=0$,

which is not satisfied unless we take a second step and substitute 0 for

$$z+ax+a^2y.$$

41. If we put $xy+yz+zx=u$ and $x^2+y^2+z^2=v$, then $\phi(u, v)=0$ is the general solution of (B). Now $x+y+z$ can be expressed as a function of u and v, for it is $+\sqrt{v+2u}$. But $z+ax+a^2y$ cannot be so expressed. It is indeed a factor of $v-u$, but that is not the proposition contended for (and purporting to be demonstrated in two distinct ways) by Boole. The general proposition (if such there be) on the subject is this: viz., that, if $u=a$ and $v=b$ be two integrals of $Pp+Qq=R$, and $w=0$ be any other solution of the equation, then w can be represented as a function of u and v, or, failing that, as a factor of such a function. In the first case we have a solution of one step; in the second we have a solution of two steps.

42. Recurring to the equations of Art. 39, viz., $\Delta\theta=0$ and $\nabla\theta=0$, let us examine the consequences of supposing that they are soluble by two steps. Put $\Delta\theta=A$ and $\nabla\theta=B$,

where A and B are taken to vanish with θ. Operate on the first with ∇, and on the second with Δ, and we have, on subtracting the results,

$$\square\,\frac{d\theta}{dz}=\nabla A-\Delta B.$$

Now, though A and B each vanish with θ, it does not follow that $\nabla A-\Delta B$ will do so. Should it however so vanish, the case may correspond with that of the discriminoidal solution $\square=0$. Should $A=0$ and $B=0$, make $\nabla A-\Delta B$ infinite, the case may correspond with $\square=\infty$. Should the given relations leave $\nabla A-\Delta B$ finite, we are thrown back on the method of the section last preceding. But the existence of solutions of two steps shows that there is no necessary conflict between the two modes of viewing the question presented in this and that section respectively.

§ 7. *On Quaternaries.*

43. Although we cannot employ the whole doctrine of determinants in the present inquiries, yet we may, I think, shorten our labours by adopting to some extent their notation. Take

$$\begin{vmatrix} a, & a_2, & a_3 \\ b, & b_2, & b_3 \\ c, & c_2, & c_3 \end{vmatrix}$$

and say that it is formed on the line when it is written $aA+a_2A_2+a_3A_3$, and on the column when it is written $aA+bB+cC$. Say, moreover, that in forming a determinant we always place the symbols in the order of the columns. Then the discriminoid

$$\square_2(P, Q, R) \quad \text{or} \quad P\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)$$

can be represented by

$$\begin{vmatrix} P, & -\frac{d}{dx}, & P \\ Q, & -\frac{d}{dy}, & Q \\ R, & -\frac{d}{dz}, & R \end{vmatrix}$$

this quasi-determinant being formed on the column and the symbols being placed in the order of columns. Proceed a step further, and write

$$\begin{vmatrix} P, & P, & -\frac{d}{dx}, & P \\ Q, & Q, & -\frac{d}{dy}, & Q \\ R, & R, & -\frac{d}{dz}, & R \\ T, & T, & -\frac{d}{dt}, & T \end{vmatrix}$$

then, the two columns on the hither side of the column of differentials being identical, we may, in conformity with the ordinary rule, suppose the result to vanish. But, developing this last quasi-determinant, we have

$$P\begin{vmatrix} Q, & -\frac{d}{dy}, & Q \\ R, & -\frac{d}{dz}, & R \\ T, & -\frac{d}{dt}, & T \end{vmatrix} - Q\begin{vmatrix} R, & -\frac{d}{dz}, & R \\ T, & -\frac{d}{dt}, & T \\ P, & -\frac{d}{dx}, & P \end{vmatrix}$$

$$+R\begin{vmatrix} T, & -\frac{d}{dt}, & T \\ P, & -\frac{d}{dx}, & P \\ Q, & -\frac{d}{dy}, & Q \end{vmatrix} - T\begin{vmatrix} P, & -\frac{d}{dx}, & P \\ Q, & -\frac{d}{dy}, & Q \\ R, & -\frac{d}{dz}, & R \end{vmatrix} = 0.$$

44. But this expresses the fact that, if we take the quaternary

$$Pdx+Qdy+Rdz+Tdt,$$

and express the four discriminoids formed by supposing x, y, z, and t successively to be constant [viz., $\square_2(Q, R, T)$, $\square_2(R, T, P)$, $\square_2(T, P, Q)$, and $\square_2(P, Q, R)$], then those four discriminoids are connected by a linear relation, so that, if three vanish identically, so will the fourth. Now, in order to apply the contents of this paper to the ascertaining whether a given quaternary admits of a single solution containing all the variables, we proceed thus: form $\square_2(P, Q, R)$, and if it vanishes identically form $\square_2(Q, R, T)$, and if that again is null form $\square_2(R, T, P)$. If that too vanishes identically, then we may be sure that $\square_2(T, P, Q)$ also vanishes identically, and that the quaternary is completely integrable. If, however, one of the discriminoids, say $\square_2(P, Q, R)$, does not vanish identically, then seek all the exceptional solutions, discriminoidal or other, of $Pdx+Qdy+Rdz$, which contain x, y, z, and t (*i.e.*, all the four variables). If no such solution can be found, the quaternary has no single solution in all the variables. If such a solution can be found, and reduces to zero the quaternary as well as the ternary, it is the required single solution of the latter. If it does not reduce the quaternary to zero, then the quaternary has no single solution whatever. It should be remarked that any one of the contained ternaries may possibly have more suitable solutions than one, and each one of the solutions of the ternary first tried should be tested by the quaternary. The process is much the same for quinary and higher forms.

Discussion of Two Double Series arising from the Number of Terms in Determinants of Certain Forms. By J. D. H. Dickson, M.A.

[*Read March 13th*, 1879.]

The first double series arises from the number of non-vanishing terms of a determinant of n^2 elements, with one diagonal of r zero-elements.

If $u_{n,r}$ be the number of such terms in a determinant as above described, it is found, by summation, in two different ways, that

$$u_{n,r} = (n-r)\,u_{n-1,r-1} + (r-1)\,u_{n-1,r-2} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

and

$$u_{n,r} = u_{n,r+1} + u_{n-1,r} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

From (2), if E, F be two operators operating only on u, and such that E refers to n alone, and F to r alone, then

$$E = EF + 1 \quad \ldots\ldots\ldots\ldots\ldots (3);$$

whence, in combination with (1), various formulæ may be obtained. For example, a formula connecting three determinants of the same order, is

$$(n-r+1)\,u_{n,r} = (n-2r+1)\,u_{n,r-1} + (r-1)\,u_{n,r-2} \quad \ldots\ldots\ (4);$$

and one connecting three consecutive numbers on the same diagonal of Table No. 1, is

$$u_{n,r} = (n-1)\,u_{n-1,r-1} + (r-1)\,u_{n-2,r-2} \quad \ldots\ldots\ (5).$$

From (5), by putting $r=n$, we have

$$u_{n,n} = (n-1)\,(u_{n-1,n-1} + u_{n-2,n-2}) \quad \ldots\ldots\ (6);$$

a first integral of which is

$$u_{n,n} = n\,u_{n-1,n-1} + (-1)^n \quad \ldots\ldots\ (7).$$

The generating function of (7) is easily found to be

$$u = \frac{e^{-x}}{1-x} \quad \ldots\ldots\ (8).$$

Table No 1 is a table of values of $u_{n,r}$, calculated by means of (2); from which, in the case of $n=7$, $r=3$, we have, as examples of (1), (2), (4), and (7),

$$\begin{aligned} 3216 &= 4.504 + 2.600, \\ 3216 &= 2790 + 426, \\ 5.3216 &= 2.3720 + 2.4320, \\ 1854 &= 7.265 - 1. \end{aligned}$$

The second double series arises from the number of non-vanishing terms of a determinant of n^2 elements, with two adjacent diagonals of r and $r-1$ zero-elements.

If $v_{n,r}$ be the number of such terms in a determinant as above described, it is found that

$$v_{n,r} = v_{n,r+1} + 2v_{n-1,r} + v_{n-2,r-1} \quad \ldots\ldots\ (9).$$

From (9), using the same operators as in the previous case, we have

$$E^2F = E^2F^2 + 2EF + 1 \text{ or } = (EF+1)^2 \quad \ldots\ldots\ (10).$$

By a process of some length, which does not appear to admit of much simplification, it is also found that

$$v_{n,n} = (n-1)(v_{n-1,n-1} + v_{n-2,n-2}) + v_{n-3,n-3} \quad \ldots\ldots\ (11),$$

a first integral of which is

$$(n-1)\,v_{n,n} = (n^2-n-1)\,v_{n-1,n-1} + n\,v_{n-2,n-2} - (-1)^n 2 \ldots (12).$$

Table No. 2 is a table of values of $v_{n,r}$, calculated by means of (9); from which, in the case of $n=8$, $r=5$, we have, as examples of (9), (11), and (12),

$$\begin{aligned} 11274 &= 8756 + 2.1168 + 182, \\ 5413 &= 7\,(675+96) + 16, \\ 7.5413 &= 55.675 + 8.96 - 2. \end{aligned}$$

TABLE No. 1. VALUES OF $u_{n,r}$, AS FAR AS $n=10$, $r=10$.

$r=$	0	1	2	3	4	5	6	7	8	9	10
n											
0	1										
1	1	0									
2	2	1	1								
3	6	4	3	2							
4	24	18	14	11	9						
5	120	96	78	64	53	44					
6	720	600	504	426	362	309	265				
7	5040	4320	3720	3216	2790	2428	2119	1854			
8	40320	35280	30960	27240	24024	21234	18806	16687	14833		
9	362880	322560	287280	256320	229080	205056	183822	165016	148329	133496	
10	3628800	3265920	2943360	2656080	2399760	2170680	1965624	1781802	1616786	1468457	1334961

TABLE No. 2. VALUES OF $v_{n,r}$, AS FAR AS $n=10$, $r=10$.

$r=$	0	1	2	3	4	5	6	7	8	9	10
n											
0	1										
1	1	0									
2	2	1	0								
3	6	4	1	1							
4	24	18	8	5	3						
5	120	96	54	34	23	16					
6	720	600	384	258	182	131	96				
7	5040	4320	3000	2136	1566	1168	883	675			
8	40320	35280	25920	19320	14664	11274	8756	6859	5413		
9	362880	322560	246960	190800	149160	117696	93582	74902	60301	48800	
10	3628800	3265920	2580480	2051280	1643760	1326120	1076064	877626	719066	591605	488592

April 10*th*, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. Donald McAlister was elected a Member; and Messrs. A. J. C. Allen, B.A., Scholar of St. Peter's College, Cambridge, and Edwyn Anthony, M.A., Christchurch, Oxford, were proposed for election.

The following communications were made:—

"Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general;" by the late Prof. Clifford, F.R.S. (communicated by Dr. Spottiswoode, P.R.S., and read in part by the Secretary).

"Short Note on the Geometrical Meaning of the Conditions for Maximum or Minimum Values of a Function of n Variables;" by Mr. J. Hammond, M.A.

"On Certain Fractions;" by Mr. R. Tucker, M.A.

The following presents were made to the Library:—

"Educational Times," April, 1879.

"On the Law of Force to any point in the plane of Motion, in order that the Orbit may be always a Conic;" by J. W. L. Glaisher, M.A., F.R.S.

"On the Law of Force to any point when the Orbit is a Conic" (and three other short Notes); from "British Association Report," 1878: by the same.

"On the Enumeration of the Primes in Burckhardt and Dase's Tables;" from "British Association Report" for 1877: by the same.

"On Multiplication by a Table of Single Entry;" from "Philosophical Magazine," November, 1878: by the same.

Various Papers and Notes that have appeared in the "Quarterly Journal of Mathematics" and the "Messenger of Mathematics" during the year 1878 (17 papers): by the same.

"Reprint of the Mathematics from the Educational Times," Vol. xxx.; from the Publishers.

"Proceedings of the Royal Society," Vol. xxviii., No. 194.

"Bulletin de la Société Mathématique de France," Tome vii., No. 3; Paris, 1879.

"Atti della Reale Accademia dei Lincei—Transunti," Vol. iii., Fasc. 4°, Marzo 1879.

"Journal de l'Ecole Polytechnique," 45^{e} Cahier, Tome xxviii.; Paris, 1878.

Astronomical and Meteorological Observations made during the year 1875 at the United States Naval Observatory, Rear-Admiral C. H. Davis, U.S.N., Vol. xxii.; Washington, 1878.

Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general. By the late Prof. Clifford, F.R.S.*

[*Read April 10th*, 1879.]

The term *alternate numbers* means a set (λ_1, λ_2) or sets of numbers which satisfy the following relations:

$$\lambda_1^2 = 0, \quad \lambda_2^2 = 0, \quad \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \ldots\ldots\ldots\ldots\ldots (1);$$

to which it is usual to add $\lambda_1\lambda_2 = 1$.

The above set is binary, but there may be ternary, &c. sets, or sets consisting of any number of letters $\lambda_1, \lambda_2, \ldots$

By a *Quadratic Form* is here meant an expression—lineo-linear in two sets of such numbers regarded as variables, say λ_1, λ_2; μ_1, μ_2, such as

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 \ldots\ldots\ldots\ldots\ldots (2).$$

This may be also denoted, for shortness, by the symbol $a \mid 12$, or even by 12, where the 1, 2 refer to the two sets of variables λ, μ.

In this case there is one invariant only which allows of the variables being separately transformed, namely, the discriminant, which is got by squaring the form. We have, in fact, by the properties of alternate numbers, $(a \mid 12)^2 = -2\,(a_{11}a_{22} - a_{12}a_{21}) = -2D$, suppose ..... (3). But if the variables are transformed by the same substitution, there is a universal covariant, $\lambda_1\mu_2 - \lambda_2\mu_1$, which may be denoted by (12), or by (21); for $\lambda_1\mu_2 - \lambda_2\mu_1 = \mu_1\lambda_2 - \mu_2\lambda_1$, by the property of alternate numbers.

If we replace the μ's by x, an ordinary, not an alternate, number, we get a linear function of the λ's, whose square vanishes in virtue of the relations (1). Thus we have $(a \mid 1x)^2 = 0$, and consequently such a product as $(a \mid 1x)\,(a \mid 1y)$ must be divisible by $(xy) = x_1y_2 - x_2y_1$

* This is the substance of some fragments found amongst the papers of the late Professor Clifford. The only published explanation of the method with which I am acquainted, is contained in the following short paragraph from a letter to Professor Sylvester (see "American Journal of Pure and Applied Mathematics," Vol. i., p. 127): "I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k = 2$, and that I have altogether 6 sets of 2 variables each, namely,

$$x_1x_2, \quad y_1y_2, \quad z_1z_2, \quad u_1u_2, \quad v_1v_2, \quad w_1w_2.$$

Suppose the forms are $(xyzu)$, $(yzvw)$, (xv), (uw);

viz., $(xyzu)$ means an expression separately linear, and homogeneous in the x, the y, the z, the u, and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables . . . To obtain this invariant, regard the variables as alternate numbers, and simply multiply all the forms together. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables."—[W. Spottiswoode.]

since it vanishes when x and y represent the same point. In fact, if we write the expressions in full, thus,

$$(a \mid 1x) = (a_{11}x_1 + a_{12}x_2)\,\lambda_1 + (a_{21}x_1 + a_{22}x_2)\,\lambda_2,$$
$$(a \mid 1y) = (a_{11}y_1 + a_{12}y_2)\,\lambda_1 + (a_{21}y_1 + a_{22}y_2)\,\lambda_2,$$

actual multiplication gives

$$(a \mid 1x)\,(a \mid 1y) = a_{11}x_1 + a_{12}x_2, \quad a_{21}x_1 + a_{22}x_2 = a_{11},\ a_{12} \times x_1,\ x_2$$
$$a_{11}y_1 + a_{12}y_2, \quad a_{21}y_1 + a_{22}y_2 \qquad a_{21},\ a_{22} \qquad y_1,\ y_2.$$

But $a_{11}a_{22} - a_{12}a_{21}$ may be regarded as the discriminant of any quadratic form having a's for its coefficients, say any form $(a \mid 1s)$. And, since it was shown, by equation (3), that the square of such a form is equal to minus twice its discriminant, it follows that the above equation may be written thus,

$$-2\,(a \mid 1x)\,(a \mid 1y) = (a \mid 1s)^2(xy) \quad \ldots\ldots\ldots\ldots \quad (4).$$

Note.—The transformation

$$y_1,\ y_2 = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} (x_1,\ x_2)$$

may also be written

$$a_{11}x_1y_1 + a_{21}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2 = 0 = (a \mid yx),$$

where the y's are arbitrary functions satisfying identically

$$y_1y_1 + y_2y_2 = 0.$$

In like manner, a second transformation

$$z_1,\ z_2 = \begin{vmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{vmatrix} (y_1,\ y_2)$$

may be represented by $(b \mid zy) = 0$; and the result of the two by

$$(b \mid zy)\,(a \mid yx) = 0.$$

There are one or two other formulæ which it will be convenient to notice before proceeding further. If we multiply the form $(a \mid 12)$ by (12), we get

$$(12)\,(a \mid 12) = a_{21} - a_{12} = s \text{ (suppose)} \quad \ldots\ldots\ldots\ldots \quad (5),$$

also, if we suppose the coefficients to remain unaltered, viz., if $(a \mid 21)$ represents $a_{11}\mu_1\lambda_1 + a_{12}\mu_1\lambda_2 + \ldots$, then

$$(a \mid 12) + (a \mid 21) = -s\,(12) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

or

$$2\,(a \mid 12) = (a \mid 12) - (a \mid 21) - s\,(12),$$

the quantity s vanishing when the form is symmetrical, *i.e.*, when $a_{12} = a_{21}$. Again, if the coefficients be supposed to remain unaltered, so that $(a \mid 13) = a_{11}\lambda_1\nu_1 + a_{12}\lambda_1\nu_2 + \ldots$, then it will be found that

$$\left.\begin{aligned}(12)\,(a \mid 23) &= -(a \mid 13) \\ (13)\,(a \mid 12) &= -(a \mid 32)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

Such multiplication is in fact tantamount to a substitution of variables, from 2 to 1 (*i.e.*, from μ to λ), or from 1 to 3 (*i.e.*, from λ to ν). This theorem, as will readily be seen, is not restricted to quadric forms; but if a, b be any two different forms, and if 1 be a set of variables in a and not in b, and 2 a set of variables in b and not in a, then, making the same supposition as in b, viz., that the constants a, b remain the same on both sides of the equation, we shall find that

$$(12)\,(a \mid 1 \ldots)\,(b \mid 2 \ldots) = (a \mid 2 \ldots)\,(b \mid 2 \ldots) = (a \mid 1 \ldots)\,(b \mid 1 \ldots).$$

With reference to the universal covariant $(12) = \lambda_1\mu_2 - \lambda_2\mu_1$, it may be remarked that we may consider such covariants for more than two sets of variables; and we shall then obtain the following formulæ: *

$$(12)\,(23) = -(13), \quad (12)\,(23)\,(34) = (14), \ldots \quad \ldots\ldots(7),$$

of which the following is a consequence,

$$(12)\,(23)\,(31) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

It is clear that quadratic forms can only be combined in a *chain*,† which, when open, gives a quadratic covariant; when closed, an invariant. We can now show that a closed chain of $2n$ sides is equal to $\pm 2D^n$, while a chain of an odd number of sides vanishes.

Take a chain of four sides, (12), (23), (34), (41); then, since

$$(12)\,(23)\,(34)\,(41) = (12)\,(23) \times (34)\,(41);$$

and since, by (3) and (4),

$$2\,(12)\,(23) = (a \mid 2_\theta)^2\,(31) = -2D\,(31),$$
$$2\,(34)\,(41) = (a \mid 4_\theta)^2\,(13) = -2D\,(13),$$
$$(31)\,(13) = 2,$$

it follows that $(12)\,(23)\,(34)\,(41) = 2D^2 \ldots\ldots\ldots\ldots\ldots\ldots (9).$

In like manner any chain of an even number of sides may be resolved into a power of the discriminant multiplied by a chain of determinants of the alternate variables. The product of these latter is ± 2, according as the number of terms is even or odd. Hence, generally, a chain of $2n$ sides is $2\,(-D)^n$.

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* This and other parts of the present paper may be compared with Spottiswoode "On Determinants of Alternate Numbers," "Proceedings of the London Mathematical Society," Vol. vii., p. 100.

† I have, in this paragraph, retained the language of the original MS., although the term *chain* is not here explained. The author appears to have had in his mind a theory which he propounded verbally at the meeting of the British Association at Bristol, and to which, in his latter days, he attached great importance and devoted much time. Some indications of it will be found in the "American Journal of Pure and Applied Mathematics," Vol. i., p. 127. Several notes relating to the subject have been found amongst his papers, but as they are almost exclusively memoranda without explanation, it is still uncertain whether they can be published.

A chain of $2n+1$ sides may, by analogous processes, be reduced to the product of a determinant of alternate variables by a form containing those variables, and must therefore vanish for symmetrical forms. For unsymmetrical it contains s as a factor. In this case the fundamental formula must be modified as follows:

$$(a \mid x1)\,(a \mid 1y) = (a \mid xy)\,s - (xy)\,D;$$

or, with alternate numbers in the place of x, y,

$$(a \mid 12)\,(a \mid 23) = (a \mid 13)\,s - (13)\,D.$$

Multiplying now into $(a \mid 31)$, we get

$$(a \mid 12)\,(a \mid 23)\,(a \mid 31) = (a \mid 13)\,(a \mid 31)\,s + sD$$

in virtue of equation (5).

But
$$\begin{aligned}(a \mid 13)\,(a \mid 31) &= (a_{11}\lambda_1\mu_1 + \ldots)\,(a_{11}\mu_1\lambda_1 + \ldots)\\ &= (2a_{11}a_{22} - a_{12}^2 - a_{21}^2) = 2D - s^2.\end{aligned}$$

Hence
$$(a \mid 12)\,(a \mid 23)\,(a \mid 31) = (3D - s^2)\,s \quad \ldots\ldots\ldots\ldots (10).$$

Again,
$$\begin{aligned}(a \mid 12)(a \mid 23)(a \mid 34) &= (a \mid 13)\,(a \mid 14)\,s - (13)\,(a \mid 34)\,D\\ &= (a \mid 14)\,s^2 - (14)\,sD + (a \mid 14)\,D\\ &= (a \mid 14)\,(D + s^2) - (14)\,sD;\end{aligned}$$

whence also
$$(a \mid 12)\,(a \mid 23)\,(a \mid 34)\,(a \mid 41) = (2D - s^2)\,(D + s^2) + s^2D \ldots\ldots (11).$$

Suppose, in general, that

$$(a \mid 12)\,(a \mid 23) \ldots (a \mid lm) = A_m\,(a \mid 1m) - B_m\,(1m);$$

then
$$\begin{aligned}(a \mid 12)\,(a \mid 23) \ldots (a \mid mn) &= A_m\,(a \mid 1m)\,(amn) + B_m\,(a \mid 1n)\\ &= A_m\{(a \mid 1n)\,s - (1n)\,D\} + B_m\,(a \mid 1n)\\ &= (sA_m + B_m)\,(a \mid 1n) - A_m D\,(1n).\end{aligned}$$

Hence
$$A_{m+1} = sA_m + B_m, \quad B_{m+1} = A_m D;$$

and consequently
$$A_{m+1} = sA_m + DA_{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

The form $\lambda_1\mu_1 + 0\lambda_1\mu_2 + s\lambda_2\mu_1 + D\lambda_2\mu_2$ is a form having s and D for invariants; and it may therefore be taken as the canonical form for a bipartite quantic such as we have been here considering. Again, another useful form is the following

$$\sqrt{(s^2 - 4D)}\,(\lambda_1\mu_2 + \lambda_2\mu_1) + s\,(\lambda\mu).$$

We may, however, combine not only two quadratic forms having like coefficients, say, two forms a; but we may also combine two having different coefficients, say, two forms a and b. Two such forms give

rise to an invariant, namely, their product. In fact, we have

$$-(a \mid 12)(b \mid 12) = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = D_{ab}, \text{ suppose};$$

and from (5), $(a \mid 12)(b \mid 12) + (a \mid 21)(b \mid 12) = -s_a s_b$,

or
$$(a \mid 21)(b \mid 12) = D_{ab} - s_a s_b \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The product of $(a \mid 12) = a_{11}\lambda_1\mu_1 + \ldots$ and $(b \mid 13) = b_{11}\lambda_1\nu_1 + \ldots$ gives a covariant; namely,

$$\begin{aligned}(a \mid 12)(b \mid 13) = &\begin{vmatrix} a_{11}\mu_1 + a_{12}\mu_2, & a_{21}\mu_1 + a_{22}\mu_2 \\ b_{11}\nu_1 + b_{12}\nu_2, & b_{21}\nu_1 + b_{22}\nu_2 \end{vmatrix} \\ = &\begin{vmatrix} a_{11}b_{21} - a_{21}b_{11}, & a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11}, & a_{12}b_{22} - a_{22}b_{12} \end{vmatrix} (\mu_1, \mu_2)(\nu_1, \nu_2) \\ = &\ \vartheta_{ab} \mid 23, \text{ suppose} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).\end{aligned}$$

If, in the covariant, we make the forms a and b coincident, it becomes

$$(a \mid 12)(a \mid 13) = D_a(23), \quad (a \mid 12)(a \mid 23) = s_a(a \mid 13) - D_a(13);$$

and, multiplying this by, or into, b, we get

$$(a \mid 12)(a \mid 13)(b \mid 43) = (b \mid 43)(a \mid 12)(a \mid 13) = -D_a(b \mid 42),$$

$$(a \mid 12)(a \mid 13)(b \mid 34) = (b \mid 43)(a \mid 12)(a \mid 13) = -D_a(b \mid 24),$$

and
$$(a \mid 12)(a \mid 13)(b \mid 23) = D_a s_b,$$

$$(a \mid 12)(a \mid 13)(b \mid 32) = D_a s_b \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The remaining form to be investigated is $(a \mid 12)(b \mid 13)(a \mid 43)$. The value of this is

$$\begin{aligned}&(a_{11}b_{21} - a_{21}b_{11})\mu_1 + (a_{12}b_{21} - a_{22}b_{11})\mu_2, \quad (a_{11}b_{22} - a_{21}b_{12})\mu_1 + (a_{12}b_{22} - a_{22}b_{12})\mu_2 \\ &\qquad a_{11}\rho_1 + \qquad\qquad a_{21}\rho_2, \qquad\qquad a_{12}\rho_1 + \qquad\qquad a_{22}\rho_2 \\ &= \ (-a_{21}a_{12}b_{11} + a_{11}a_{21}b_{12} + a_{11}a_{12}b_{21} - a_{11}a_{11}b_{22})\,\mu_1\rho_1 \\ &\quad + (-a_{21}a_{22}b_{11} + a_{21}a_{21}b_{12} + a_{11}a_{22}b_{21} - a_{11}a_{21}b_{22})\,\mu_1\rho_2 \\ &\quad + (\ a_{12}a_{12}b_{21} - a_{22}a_{12}b_{21} - a_{11}a_{12}b_{22} + a_{11}a_{22}b_{12})\,\mu_2\rho_1 \\ &\quad + (\ a_{22}a_{12}b_{21} - a_{22}a_{22}b_{11} - a_{21}a_{12}b_{22} + a_{21}a_{22}b_{12})\,\mu_2\rho_2 \\ &= \ (-a_{11}D_{ab} + b_{11}D_{aa})\,\mu_1\rho_1 \\ &\quad + (-a_{21}D_{ab} + b_{21}D_{aa})\,\mu_1\rho_2 \\ &\quad + (-a_{12}D_{ab} + b_{12}D_{aa})\,\mu_2\rho_1 \\ &\quad + (-a_{22}D_{ab} + b_{22}D_{aa})\,\mu_2\rho_2 \\ &= D_{ab}(a \mid 42) - D_{aa}(b \mid 42) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).\end{aligned}$$

Multiplying this into $(b \mid 45)$, we obtain

$$\begin{aligned}&(a \mid 12)(b \mid 13)(a \mid 43)(b \mid 45) \\ &\qquad = D_{ab}(a \mid 42)(b \mid 45) - D_{aa}(b \mid 42)(b \mid 45);\end{aligned}$$

that is, referring to equation (14), and dropping the suffix ab,

$$(\vartheta \mid 23)(\vartheta \mid 35) = D_{ab}(\vartheta \mid 25) - D_{aa} \cdot D_{bb}(25).$$

But $$(\vartheta \mid 23)(\vartheta \mid 35) = s(\vartheta_\vartheta \mid 25) - D_\vartheta(25).$$

Hence $$s_\vartheta = D_{ab}, \quad D_\vartheta = D_{aa} \cdot D_{bb} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

as may be easily verified.

If, however, we multiply the equation $(a \mid 12) + (a \mid 21) = -s(12)$ by $(a \mid 12)$, we obtain

$$(a \mid 12)(a \mid 21) = 2D_a - s_a^2;$$

hence $$(\vartheta \mid 12)(\vartheta \mid 21) = 2D_\vartheta - s_\vartheta^2 = 2D_aD_b - D_{ab}^2 \ldots\ldots\ldots\ldots (18),$$

and consequently

$$\left.\begin{aligned} \{(\vartheta \mid 12) - (\vartheta \mid 21)\}^2 &= -4D_aD_b - 2(2D_aD_b - D_{ab}^2) \\ &= -2(4D_aD_b - D_{ab}^2) \end{aligned}\right\} \ldots\ldots (19).$$

The formula $s_\vartheta = D_{ab}$ is important as showing that ϑ is not made symmetrical by making a and b symmetrical. Hence, in passing from these invariants and covariants to those of symmetrical forms, we are obliged to use the symmetrised form of ϑ, namely $(\vartheta \mid 12) - (\vartheta \mid 21)$. Thus, to adapt equation (16) to symmetrical forms, we have

$$\begin{aligned} (\vartheta \mid 23)(a \mid 43) &= D_{ab}(a \mid 42) - D_{aa}(b \mid 42), \\ (\vartheta \mid 23)(a \mid 43) &= (a \mid 13)(b \mid 12)(a \mid 43) \\ &= D_a(14)(b \mid 12) = -D_a(b \mid 42). \end{aligned}$$

In general, we write $\bar{a}$ for the mean value of a, *i.e.*, if

$$2(\bar{a} \mid 12) = (a \mid 12) - (a \mid 21),$$

we shall have $$(\bar{a} \mid 12) = (a \mid 12) + \tfrac{1}{2}s(12).$$

Hence, since $(12)^2 = -2$,

$$(\bar{a} \mid 12)^2 = (a \mid 12)^2 + s^2 - \tfrac{1}{2}s^2 \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and $$\overline{D} = D - \tfrac{1}{4}s^2,$$

where $\overline{D}$ is the discriminant of the symmetrical function.

Similarly

$$\begin{aligned} \vartheta_{\bar{a}\bar{b}} = (\bar{a} \mid 12)(\bar{b} \mid 13) &= \{(a \mid 12) + \tfrac{1}{2}s_a(12)\}\{(b \mid 13) + \tfrac{1}{2}s_b(13)\} \\ &= \vartheta_{ab} - \tfrac{1}{2}\{s_a(b \mid 23) + s_b(a \mid 23)\} + \tfrac{1}{4}s_as_b(23), \end{aligned}$$

and therefore

$$\left.\begin{aligned} 2\overline{\vartheta}_{\bar{a}\bar{b}} &= (\vartheta_{ab} \mid 23) - (\vartheta_{ab} \mid 32) - s_as_b(23) \\ &= 2(\vartheta_{ab} \mid 23) - s_\vartheta(23) - s_as_b(23) \end{aligned}\right\} \ldots\ldots\ldots\ldots (21).$$

If we multiply this into $(a \mid 43)$, we have

$$\begin{aligned} 2(\vartheta_{\bar{a}\bar{b}} \mid 23)(a \mid 43) &= 2(\vartheta \mid 23)(a \mid 43) - D_{ab}(a \mid 42) - s_as_b(a \mid 42) \\ &= D_{ab}(a \mid 42) - 2D_aD_b(b \mid 42) - s_as_b(a \mid 42). \end{aligned}$$

Putting $s_a = 0$, $s_b = 0$, we get the formula for symmetric functions.

May 8th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Messrs. A. J. C. Allen and Edwyn Anthony were elected Members; and Messrs. Allen, Hargreaves, and McAlister were admitted into the Society.

Dr. Hirst, F.R.S., presented a carte-de-visite likeness of M. Chasles to the Society's Album; and Mr. Tucker, on behalf of Mrs. Clifford, presented a likeness of the late Prof. Clifford to take the place of the earlier likeness in the Album.

The following communications were made:—

"Note on the Complexes generated by two Correlative Planes:" Dr. Hirst, F.R.S.

"On a Theorem relating to Conformable Figures:" by Prof. Cayley, F.R.S.

"Notes on some Definite Integrals:" by late Prof. Clifford (communicated by Mr. Tucker).

"A Method of Constructing by Pure Analysis Functions X, Y, &c., which possess the property that $\int XYd\sigma = 0$, and such that any given function can be expanded in the form $\alpha X+\beta Y+\gamma Z+\ldots$:" by Mr. Routh, F.R.S.

"The Numerical Calculation of a Class of Determinants, and a Continued Fraction:" by Mr. J. D. H. Dickson, M.A.

"The Nephroid, Heptagon, &c.:" by Rev. Dr. Freeth.

The following presents were received:—

(Second Memoir) "On the Equations of Circles:" by Dr. Casey, F.R.S. ("Trans. of Royal Irish Academy," Vol. xxvi.): from the Author.

"Educational Times," May, 1879.

"Monatsbericht," Januar, Februar, 1879.

"Proceedings of Royal Society," Vol. xxviii., No. 195.

"Journal of Institute of Actuaries," Vol. xxi., Pt. iv., No. cxiv., Jan., 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome ii., Dec. 1878.—"Table des Matières, et Noms d'Auteurs," Tome ii. (xiii. de la collection); Tome iii. Janv. 1879 (xiv. de la collection).

"Atti della R. Accademia dei Lincei—Transunti," Vol. iii., Fasc. 5°, Apr. 1879.

"Archiv for Mathematik og Naturvidenskab," Første Bind, Fjerde Hefte; Andet Bind, Første, Andet, Tredie, Fjerde Hefte; Tredie Bind, Første, Andet, Tredie Hefte.

Note on the Complexes generated by Two Correlative Planes.

By T. Archer Hirst, F.R.S.

[*Read May 15th*, 1879.]

1. When a general correlation* has been established between two planes α and β, arbitrarily situated in space, each point A in the former is said to be *conjugate* to every point of its corresponding line b in the latter. The number of pairs of conjugate points in α and β, therefore, is triply infinite, and the lines c which pass through the points of the several pairs form a complex **C** which may be said to be *generated* by the given correlative planes or, more briefly, by the given correlation.

2. The lines which correspond to the several points of the intersection $\overline{\alpha\beta}$ of the given planes form two homographic pencils whose centres A_0 and B_0 are the points corresponding to that intersection, regarded, successively, as a line of β and a line of α. *Every line situated in either of the given planes α or β belongs to the complex* **C**, since it joins a point on $\overline{\alpha\beta}$ to one of its conjugate points. Moreover, two of the points on $\overline{\alpha\beta}$, and in general only two, will lie in their corresponding lines, and therefore be *self-conjugate*. *Every line in space which passes through either of the two self-conjugate points C_1 and C_2 also belongs to the complex* **C**†.

The correlated planes α, β might, it is true, be so placed that more than two points would become self-conjugate; should this be the case, however, every point in $\overline{\alpha\beta}$ would be self-conjugate, and the complex **C** would degenerate to two linear complexes.

3. The complex-lines passing through any point A of α form two pencils; one is in the plane α, the other in the plane of A and its corresponding line b. In other words, the complex-cone at A breaks up into a plane-pair‡; whence we at once infer—first, that *the complex* **C** *is of the second degree;* secondly, that *every point of α and β is a singular point of the complex;* and thirdly, that *every plane through a point and its corresponding line is a singular plane of the complex.* The complex-lines in (Ab), for instance, form two pencils (A) and (B),

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* See my paper,—"On the Correlation of Two Planes," "Proceedings of the London Mathematical Society," 1874, Vol. v., p. 40.

† Plücker, in his "Neue Geometrie des Raumes," 1868, p. 309, terms points and planes possessing the properties of C_1, C_2, α, β the *ausgezeichnete* points and planes of the complex.

‡ The planes of the plane-pair coincide in α when A corresponds to $\overline{\alpha\beta}$. According to Plücker, therefore (*loc. cit.*, p. 308), A_0 and B_0 are *double points* of the complex **C**.

where B is the point, in b, which corresponds to the intersection a of a and the plane (Ab). *The complex-conic*, therefore, *in the plane of any two conjugate lines a, b, which cut*, or are incident with, *one another, breaks up into the pair of points B, A corresponding to those lines.*

4. By a well-known theorem,* the plane of a point and its corresponding line, or, as it may otherwise and more conveniently be defined, *the plane of two conjugate and incident lines of two correlative planes envelopes a surface of the second class.*

We will denote the quadric thus *generated* by the given correlation in α and β by S^2; it clearly touches these planes in the points A_0 and B_0, and passes through the two, real or imaginary, self-conjugate points C_1, C_2; in fact, it contains each of the four lines $\overline{A_0C_1}$, $\overline{A_0C_2}$, $\overline{B_0C_1}$, $\overline{B_0C_2}$ since each, as corresponding to the self-conjugate point through which it passes, is obviously conjugate to every line, in the plane to which it does not belong, which is incident therewith.

5. It can be readily shown that the complex-conic in any arbitrary plane π is the common trace of the two cones, circumscribed to S^2, whose vertices are the points which correspond to the intersections of π with α and β. Hence it follows, not only that *every complex-conic touches the planes α and β*, but also that *it has double contact with the quadric S^2*. The common tangents of the conic and surface are the traces of the common tangent planes of the above cones, and they are the only two lines in π that connect points in α and β, whose corresponding lines are both conjugate to, and incident with one another.

6. There are two, and only two, cases in which *a complex-conic breaks up into a point-pair.*

The first is *when its plane π touches the quadric S^2*. The two common tangents, above referred to, then coincide in the connector of the points of the complex-point-pair (Art. 3); whence it follows that *every such connector is a tangent to S^2.*

The second case is *when its plane π passes through one of the self-conjugate points C_1, C_2*. The vertices of the two circumscribed cones referred to in Art. 5 then fall on the generators of S^2, which, as we have seen in Art. 4, pass through that self-conjugate point. Each of the cones in question, therefore, degenerates to a line-pair, formed by two generators of S^2, and their common trace on π breaks up into a point-pair, whereof one constituent is the self-conjugate point through which π passes.

7. The points of the point-pair coincide, and their connector becomes indeterminate in direction, when π is one of the tangent planes γ_1, γ_2 of

* First given by Seydewitz in Grunert's "Archiv," 1847, Bd. ix., p. 193. See also Reye, "Geometrie der Lage" (zweite Abtheilung), 1868, p. 26.

S^2 at the self-conjugate points C_1, C_2. *Each of these planes* γ_1, γ_2 *is a double plane* of the complex* **C**.

8. When the points and corresponding lines of the planes α, β are projected from a point P, situated in neither plane, two concentric and correlative pencils are obtained. To each line p [plane π] thereof corresponds one or other of two planes π_1, π_2 [lines p_1, p_2], according as the former is conceived to project a point [line] of β or of α.

Should a line p lie in either of its two corresponding planes π_1, π_2, it will necessarily lie in both, and be a *complex-line*. Should, on the other hand, a plane π pass through either of its corresponding lines p_1, p_2, it will necessarily pass through both, and be a *tangent-plane* of the quadric S^2. *The complex-cone, and the circumscribed cone, at any point* P, *are thus seen to be correlative figures;* the generators of the former and tangent planes of the latter being associated in two different ways, but always so that their associated elements are mutually incident; in other words, so that each tangent plane of the latter passes through the generator of the former with which it is associated.

Every complex-cone passes through the self-conjugate points C_1, C_2.

9. From the doubly correlative relation just indicated it follows, first, that *the complex- and circumscribed cones at any point* P *have double contact with each other*, the two generators of contact (tangents to S^2) being the only two lines, passing through P, which connect points in α and β, whose corresponding lines in β and α are both incident and conjugate (Art. 5); and, secondly, that *the degeneration of one of the two cones, necessarily involves that of the other.*

10. From the second of the above properties we at once infer, that *the singular points of the complex* **C**, *excluding those situated in the planes* α, β (Art. 3), *all lie on the quadric* S^2. At every point P on S^2, in fact, and at these only, the circumscribed cone degenerates to a line-pair, and the complex-cone to a plane-pair. The lines of the former are generators of S^2, and the tangent plane of S^2 which contains them, corresponds to, and therefore passes through (Art. 8), the intersection of the planes of the latter.

With this intersection coincide, of course, the two generators along which the complex- and circumscribed cones ordinarily touch each other; it passes, therefore, through the points that correspond to the lines in which the tangent plane at P intersects α and β (Art. 9), and consequently connects the pair of points to which the complex-conic in that plane degenerates (Art. 6).

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* Plücker, *loc. cit.*, p. 308.

11. Hence, and from Arts. 3, 6, and 7, it follows that *the singular surface of the complex* **C** *breaks up into two surfaces of the second order and class; one of these is the quadric* S^2, *which the given correlation itself generates; the other breaks up into the generating planes* α, β, *together with the self-conjugate points* C_1, C_2.

12. *The singular lines of the complex* **C** *form a congruence* (4, 4) *which breaks up into two congruences* (1, 0), *two congruences* (0, 1), *and a congruence* (2, 2).

The lines of each of the congruences (1, 0) pass through a self-conjugate point. Each such line connects the points of a complex-point-pair, situated in a plane which is, in general, uniquely determined (Art. 6). The complex-plane-pair, however, whose planes intersect in the line in question, is indeterminate, every line through a self-conjugate point being a complex-line (Art. 2).

The lines of the two congruences (0, 1) fill the generating planes α, β. In each such line the two planes intersect that constitute the complex-cone at a point thereof, which is in general uniquely determined (Art. 3). The complex-point-pair, however, whose points the line in question connects, is indeterminate, since every line in the generating planes is a complex-line (Art. 2).

Each line of the congruence (2, 2) *passes through a pair of points in* α, β *to which conjugate and incident lines correspond;* it is, therefore, the connector of the points of the complex-point-pair situated in one of the tangent planes of S^2, as well as the intersections of the complex-plane-pair at the point of contact of that tangent plane.

13. By virtue of the property last referred to, the congruence (2, 2) may be said to consist of the *proper singular lines* of the complex **C**. It is in general irreducible, and will henceforth be denoted by **G**. The quadric S^2 forms part of its focal surface, the residual constituent thereof, as will be seen in Art. 21, being another quadric S^2. This congruence **G**, moreover, includes four plane pencils. The centres of two are the double points A_0, B_0 (Art. 3), and the planes of the remaining two are the double planes γ_1, γ_2 (Art. 7). The centres of the latter pencils are the self-conjugate points C_1, C_2, and the planes of the former are α, β. *The four lines* $\overline{A_0C_1}$, $\overline{A_0C_2}$, $\overline{B_0C_1}$, $\overline{B_0C_2}$, each of which belongs to two of the above pencils, *are double lines of the complex* **C**; since in every plane passing through any one of them the complex-conic degenerates to two points situated therein, and at every point of the line in question the complex-cone degenerates to a pair of planes which intersect in that line.* *A fifth double line of the complex is the intersection* $\overline{\alpha\beta}$. The complex-plane-pair at every point

* Plücker, *loc. cit.*, p. 309.

of it is (α, β); the complex-point-pair in every plane through it is (C_1, C_2)*.

Systems of Complexes in Involution.

14. Between the conjugate and incident lines a, b, of the planes α, β a quadric relation (or correspondence) exists; that is to say, to each line $a\,[b]$ corresponds, in general, one and only one line $b\,[a]$, whilst to the several lines $a\,[b]$ which pass through a point $A\,[B]$ correspond lines $b\,[a]$ which envelop a conic,—the trace, in fact, on the plane $\beta\,[\alpha]$, of the cone circumscribed to S^2 whose vertex is at $A\,[B]$. The *conics* which thus correspond to the several *points* of $\alpha\,[\beta]$ are all inscribed in the triangle $B_0C_1C_2\,[A_0C_1C_2]$. The sides and vertices of each such triangle are the *principal lines* and *principal points* of the quadric relation. It will be convenient to denote C_1 and C_2, regarded as principal points of the plane $\alpha\,[\beta]$, by A_1 and $A_2\,[B_2$ and $B_1]$ respectively, and the principal line opposite to $A_i\,[B_i]$ in the *principal triangle* $A_0A_1A_2\,[B_0B_1B_2]$ by $a_i\,[b_i]$. We will further distinguish, as *associated*, any two principal elements, in different planes, which are denoted by letters with like suffixes. To every principal line corresponds, in the quadric relation, an indeterminate line through its associated principal point; and to every point $A\,[B]$ in a principal line corresponds a conic which breaks up into the associated principal point, and a point $B\,[A]$ in the associated principal line. A and B are always corresponding points in the two homographic rows.

15. Every quadric relation between the lines of two planes represents, as I have elsewhere shown,† a one-fold system of correlations, such that each pair of corresponding lines in the quadric relation is a pair of conjugate lines in *every* correlation of the system. The quadric relation is determined when, in addition to its three pairs of associated principal lines, any two corresponding lines a and b, are given; for the complete determination of any one of the correlations of the system which this quadric relation represents, it will suffice, in addition to the above data, to give the point A in a which is to correspond to b.

16. Each pair of corresponding lines in the quadric relation of Art. 14 being a pair of incident lines, it is obvious that *one and the same quadric S^2 is generated by every correlation of the system represented by that quadric relation. These correlations, therefore, generate a one-*

* The complex **C** is identical with the one described by Weiler in Art. 13 of his valuable Memoir, "Ueber die verschiedene Gattungen der Complexe zweiten Grades" (Math. Ann. 1874, Bd. vii., p. 177). The author desires me to state, however, that there, as well as on p. 206, the singular surface of the complex is inaccurately described as breaking up into "a surface of the second degree, together with two of its points, and the tangent planes *at these points*."

† "Correlation of Two Planes," Art. 48.

fold system of complexes **C** *having a common singular surface* S^2, $(\alpha, \beta; C_1, C_2)$ (Art. 11). The several rays of the pencil of tangents at any point P of the quadric S^2 serve to individualise the several complexes of the system, since each such ray is a proper singular line of one, and only one, such complex. *The several congruences* **G**, *belonging to the system of complexes* **C**, *together make up the complex of the second degree whose lines are the tangents of* S^2, *the tangential complex of* S^2.

17. The system of complexes under consideration includes three, $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$, which break up, individually, into a pair of special linear complexes; their respective directrices are the three pairs of associated principal lines a_0, b_0; a_1, b_1; a_2, b_2; and the correlations which generate them are the three exceptional ones of the system which have these lines for *singular lines*.*

The principal lines a_0, b_0 being coincident in $\overline{\alpha\beta}$ the complex $\mathbf{C}_0$ resolves itself into two coincident special linear complexes. Regarded as a complex of the second degree having S^2, $(\alpha, \beta; C_1, C_2)$ for singular surface, *the proper singular lines of* $\mathbf{C}_0$ *form a congruence* $\mathbf{G}_0$, *which has* $\overline{\alpha\beta}$ *for focal line, and* S^2 *for focal surface*.†

The congruence $\mathbf{G}_1$ formed by the proper singular lines of the complex $\mathbf{C}_1$ breaks up, as does the complex itself, into two linear constituents. It consists, in fact, of all the tangents of S^2 whose points of contact lie on the principal lines a_1 and b_1, which latter are directrices of $\mathbf{C}_1$ as well as of S^2. The congruence $\mathbf{G}_1$, therefore, includes the system of generators of S^2 which are incident with these directrices.

The complex $\mathbf{C}_2$, and its congruence $\mathbf{G}_2$, have exactly similar properties: the directrices a_1 and b_1 being merely replaced by a_2 and b_2.

18. A fourth complex $\mathbf{C}_3$, of the system under consideration is noteworthy, from the circumstance that, although it does not degenerate like $\mathbf{C}_0$, its congruence $\mathbf{G}_3$ has, like $\mathbf{G}_0$, a focal line as well as a focal surface. This line, in the case of $\mathbf{G}_3$, passes through A_0 and B_0.

19. Between the pairs of points A, B, which correspond to conjugate and incident lines, and in which the several lines of the original congruence **G** (Art. 13) pierce the planes α, β, a quadric relation exists which is simply the correlative form of that described in Art. 14, and has, with the latter, the same principal lines and points.‡ Every such quadric relation between the points of two planes α, β represents a one-fold

* "Correlation of Two Planes," Art. 28, and "Principle of Duality."

† Kummer describes such a congruence at p. 28 of his Memoir, "Ueber die algebraischen Strahlensyteme," Abhand. d. k. Akad. d. Wissen., Berlin, 1866, p. 28

‡ The lines which join corresponding points of two planes in quadric relation with each other, as I shall have occasion to shew in a future communication, form a congruence which is, in general, of the fourth order and second class. The reduction of this order to the second, in the present instance, is due to the coincidence of two non-associated principal points in each of the points C_1, C_2.

system of correlations such that each pair of corresponding points of the quadric relation is a pair of conjugate points in *every* correlation of the system.* *The complexes generated by the several correlations of this system form a pencil*, since they have in common the congruence (4, 4) composed of the lines of the congruence **G**, of those of the self-conjugate points C_1, C_2, and of those of the generating planes α, β. The quadrics generated by the correlations of the system likewise form a pencil, since each contains the four lines a_1, a_2, b_1, b_2. *The singular surfaces*, therefore, *of the several complexes of the pencil themselves form a pencil*, each being composed of a quadric of the pencil $(a_1a_2b_1b_2)$, in conjunction with the plane- and point-pair $(\alpha, \beta; C_1, C_2)$.

20. The quadric relation between the points of α and β is determined when, in addition to the associated principal elements, any pair of corresponding points A, B thereof (or a line of the congruence **G**) is given. A correlation of the system represented by this quadric relation is determined when, in addition to the above data, the line b, through B, is given which is to correspond to A. If, in the correlation thus determined, a be the line which corresponds to B, it is obvious that (Ab) and (Ba) will be the two tangent planes, through $\overline{AB}$, to the quadric generated by the correlation in question; and since every such quadric belongs to the pencil $(a_1a_2b_1b_2)$, *(Ab) and (Ba) will be conjugate planes of two co-axal pencils in involution.*

These planes determine a second correlation of the system by which the quadric above signalised may be generated. To A and B, in this second correlation, correspond, respectively, the lines b' and a', in which β and α are intersected by the planes (Ba) and (Ab).

21. The double planes of the above involution touch, and the correlations they respectively determine generate, the only two quadrics S^2 and $\overline{S}^2$ of the pencil $(a_1a_2b_1b_2)$ which touch the line $\overline{AB}$. In each of these two correlations, and in these only, the points A, B correspond to incident and conjugate lines. The complexes **C**, $\overline{\mathbf{C}}$ which they generate are, consequently, distinguished from all others, in the pencil to which they belong, by having $\overline{AB}$, and with it every line of the congruence **G**, for a proper singular line. *The quadrics S^2 and $\overline{S}^2$ together constitute*, as already stated in Art. 13, *the focal surface of the congruence* **G**.

By a well-known theorem, the points [planes] in which each line of **G** touches S^2 and $\overline{S}^2$ are harmonic conjugates relative to the points [planes] which that line determines with α, β $[C_1, C_2]$, as well as with γ_1, γ_2 $[A_0, B_0]$.

* "Correlation of Two Planes," Art. 47.

22. From the above we conclude that *the pencil of complexes of Art.* 19 *consists of pairs in involution, the paired (conjugate) complexes being those which have for common singular surface* $(\alpha, \beta; C_1, C_2)$, *together with any quadric of the pencil* $(a_1a_2b_1b_2)$. *The lines of the congruence* **G**, *common to every complex of the pencil, are proper singular lines of each of the two complexes* **C**, $\overline{\mathbf{C}}$, *and of these solely.* The latter form, in fact, what may be termed the double complexes of the involution.

23. In the system by which the quadric relation between the points of α and β is represented (Art. 19), three exceptional correlations are included, each having a pair of associated principal points for *singular points.** The quadric generated by each is a point- and plane-pair, and the complex a *tetrahedral* one.†

24. When the singular points are A_0 and B_0, for instance, *the complex* $\mathbf{C}^0$, which the correlation generates, *has the vertices and faces of the tetrahedron* $\alpha\beta\gamma_1\gamma_2$ *for its singular surface;* ‡ *every complex-conic touches the faces of this tetrahedron, and every complex-cone passes through its vertices.*

Hence, every complex-line determines with the faces $\alpha\beta\gamma_1\gamma_2$ four points, and with the vertices $A_0B_0C_1C_2$ four planes, which have a constant anharmonic ratio. *This property is shared,* of course, *by every proper singular line of the original complex* **C**, since all the lines of the congruence **G** belong to the tetrahedral complex $\mathbf{C}^0$ (Art. 19).

25. *The complex* $\mathbf{C}_0^0$ *conjugate to* $\mathbf{C}^0$ *in the involution of Art.* 22, having with it one and the same singular surface $(\alpha, \beta; C_1, C_2)$, $(A_0, B_0; \gamma_1, \gamma_2)$, *breaks up into two linear congruences, one of which is general, and contains the congruence* **G**, *whilst the other is special and has* $\overline{\alpha\beta}$ *for directrix.*

This degeneration is due to the occurrence of the contingency referred to in Art. 2. The two correlations, in fact, which generate the conjugate complexes under consideration are determined by the pair of planes (ABA_0) and (ABB_0) of the involution described in Art. 20. If the line $b \equiv \overline{BB_0}$, in which the plane (ABB_0) cuts β, correspond to A, the correlation thereby determined is that which has A_0, B_0 for singular points, and generates the tetrahedral complex $\mathbf{C}^0$. If, on the other hand, the line b', in which β is cut by the plane (ABA_0), be taken to correspond to A, the point of intersection of b' and $\overline{\alpha\beta}$ will correspond to the line $\overline{AA_0}$ in which it lies; it will, accordingly, be the *third* self-

* "Correlation of Two Planes," Art. 28.

† Reye, *loc. cit.*, p. 116.

‡ A tetrahedral complex is thus seen to be the aggregate of all the linear congruences whose directrices are corresponding rays of any two homographic plane pencils arbitrarily situated in space.

conjugate point whose existence, as observed in Art. 2, involves that of an infinite number.

26. In each of the two remaining exceptional correlations, referred to in Art. 23, the singular points are coincident with the self-conjugate points C_1 and C_2. In one of the two the latter represent the associated principal points A_1, B_1, respectively, in the other, B_2, A_2. Both correlations generate the point- and plane-pair $(C_1, C_2; \alpha, \beta)$, so that this degenerate quadric, twice counted, constitutes the singular surface common to the complexes $\mathbf{C}'$, $\mathbf{C}''$ respectively generated by them. These complexes, which are obviously conjugate to one another in the involution of Art. 22, are special tetrahedral ones; two faces of the tetrahedron being now coincident in each of the planes α and β. The complex-cones of $\mathbf{C}'$ [$\mathbf{C}''$] all touch in A_1 and B_1 [A_2 and B_2] the principal lines a_2 and b_2 [a_1 and b_1]; its complex-conics all touch α and β at points on these lines.

27. We have seen in Art. 21, that the proper singular lines of the original complex $\mathbf{C}$ form a congruence $\mathbf{G}$, of the second order and class, whose lines are common tangents of the quadric S^2, generated by the correlation from which $\mathbf{C}$ itself proceeds, and of a second quadric $\overline{S}^2$ belonging, with S^2, to the pencil $(a_1a_2b_1b_2)$. The *complete* congruence of all such common tangents, however, is clearly of the fourth order and class; hence the latter must break up into $\mathbf{G}$ and another congruence $\dot{\mathbf{G}}$ of the second order and class.

This congruence $\dot{\mathbf{G}}$ consists of the proper singular lines of a complex $\dot{\mathbf{C}}$ which belongs, with $\mathbf{C}$, to *the system*, described in Art. 16, *having a common singular surface* S^2, $(\alpha, \beta; C_1, C_2)$; whence it follows that *the system in question may also be regarded as consisting of pairs of complexes in involution; conjugate complexes* $\mathbf{C}$, $\dot{\mathbf{C}}$ *thereof, being those whose proper singular lines form two congruences* $\mathbf{G}$, $\dot{\mathbf{G}}$, *which together include all the common tangents of* S^2 *and any other quadric* $\overline{S}^2$ *of the pencil* $(a_1a_2b_1b_2)$.

28. The double complexes of the involution just defined are the complexes $\mathbf{C}_0$ and $\mathbf{C}_3$ described in Arts. 17 and 18. The proper singular lines of the former are the common tangents of S^2 and the plane- and point-pair $(\alpha, \beta; C_1, C_2)$; those of the latter are the common tangents of S^2 and $(\gamma_1, \gamma_2; A_0, B_0)$. The two complexes $\mathbf{C}_1$, $\mathbf{C}_2$, described in Art. 17, are conjugate to one another in the involution under consideration. Their proper singular lines are to be regarded as the common tangents of S^2 and a second quadric *which has become coincident therewith.*

29. From the same Art. 21, we infer the existence of a third system of complexes in involution. It was there seen that the lines of the

congruence $\mathbf{G}$ are proper singular lines, not only of $\mathbf{C}$, but of another complex $\overline{\mathbf{C}}$ whose singular surface is $\overline{S}^2$, $(\alpha, \beta; C_1, C_2)$, and in like manner it may be shewn that the lines of the congruence $\dot{\mathbf{G}}$, referred to in Art. 27, are proper singular lines, not only of $\dot{\mathbf{C}}$, but of a second complex $\hat{\mathbf{C}}$.

The complexes $\overline{\mathbf{C}}$, $\hat{\mathbf{C}}$, *whose common singular surface is* $\overline{S}^2, (\alpha, \beta; C_1, C_2)$, *where* $\overline{S}^2$ *is any quadric of the pencil* $(a_1a_2b_1b_2)$, *are conjugate ones in the above third involution. Their proper singular lines together make up the system of tangents common to* $\overline{S}^2$ *and a fixed quadric* S^2 *of the above pencil.* This third involution has, in common with that described in Art. 27, the conjugate complexes $\mathbf{C}_1$, $\mathbf{C}_2$ of Art. 17. Its double complexes are to be regarded as very special tetrahedral ones. One, in fact, has the singular surface $(\alpha, \beta; C_1, C_2)$, counted twice, and resolves itself into the linear complex, twice counted, whose directrix is $\overline{\alpha\beta}$; the other has the singular surface $(\gamma_1, \gamma_2; A_0, B_0)$, $(\alpha, \beta; C_1, C_2)$, and breaks up into the two linear complexes whose directrices are $\overline{A_0B_0}$ and $\overline{\alpha\beta}$.

Special Positions of Generating Planes.

30. Although the correlation between α and β may retain its general character, special complexes are generated when special positions are given to these correlated planes. The three cases of most interest are—*first*, when their self-conjugate points C_1, C_2 coincide; *secondly*, when the points A_0, B_0 lie in the intersection $\overline{\alpha\beta}$ to which they correspond; and *thirdly*, when the points A_0, B_0 themselves coincide. Referring for further details to a paper about to be published* elsewhere, the principal features which present themselves in these cases shall here be briefly indicated.

31. No matter what lines a_0, b_0 may be selected for coincidence in $\overline{\alpha\beta}$, it is always possible so to place the two correlated planes α, β that their self-conjugate points C_1, C_2 shall be coincident†; and with them, of course, the principal points A_1, A_2, B_1, B_2. Let C_0 be the common point of coincidence, and let the lines $\overline{A_0C_0}$ and $\overline{B_0C_0}$, with which the principal lines a_1, a_2 and b_1, b_2 now respectively coincide, be denoted by a_{12} and b_{12}, and their plane, with which γ_1 and γ_2 now coincide, by γ_0.

The correlation being of a general kind, the quadric S^2 now generated by the correlated planes breaks up into the *doubled* plane γ_0, bounded by

* Chelini Memorial Volume.

† See Notes A. and B. to my paper on "The Degenerate Forms of Conics," "Proceedings of the London Mathematical Society," 1869, Vol. ii., p. 171.

a conic s^2 which touches a_{12} and b_{12} in A_0 and B_0. In addition to quadrics of this degenerate form, the pencil $(a_1a_2b_1b_2)$ will contain cones σ^2 which touch α and β along their common generators a_{12}, b_{12}; the correlations which generate the latter, however, are all exceptional ones, each having a pair of associated singular points coincident in C_0.

The singular surface of the complex **C** *generated by any ordinary correlation now consists of the doubled plane* γ_0, *bounded by the conic* s^2, *together with the plane-pair* α, β, *supplemented by the doubled point* C_0. *Every complex-conic is incident with, and every complex-cone touches the conic* s^2 *at two points: the former, moreover, touches the planes* α *and* β, *as usual and the latter has contact with* $\overline{\alpha\beta}$ *at* C_0. *The proper singular lines of the complex* **C** *form a congruence* **G** *which has the conic* s^2 *for focal curve, and one of the above cones* σ^2 *for focal surface.**

This congruence **G** is a very special form of the one which is thus described by Kummer in his masterly Memoir "On Algebraical Systems of Rays" †: "Excluding right lines which merely pass through the vertex of a given quadric cone, all those which touch that cone, and at the same time cut a curve of the n^{th} order, passing $n-2$ times through its vertex and itself touching the cone twice, form *two different ray-systems* (congruences) of the second order and n^{th} class."

In our case $n=2$, and the plane of the curve (conic) passes through the vertex of the cone. The second of the two congruences referred to in the above theorem is composed, in our case, of the proper singular lines of the complex $\dot{\mathbf{C}}$, conjugate to **C** in the involution of complexes, having the same singular surface, described in Art. 27.

32. The second case of Art. 30 presents itself whenever any two *conjugate lines* of two correlated planes α, β are brought, arbitrarily, into coincidence. The self-conjugate points are now the points A_0 and B_0 themselves; in fact, the principal points A_0, A_1, B_2 all coincide in C_1, and B_0, B_1. A_2 in C_2. From this it follows that the principal lines a_0, a_1, b_0. b_1 are all coincident in $\alpha\beta$, a_2 and b_2 alone retaining arbitrary positions through C_1 and C_2 respectively.

Hence we infer that the quadric S^2 generated by two such correlated planes is, in general, a hyperboloid, of which a_2 and b_2 are directrices, and $\overline{\alpha\beta}$ a generator; *this hyperboloid, and its tangent planes* α, β, *together with their points of contact* C_1, C_2, *now constitute the singular surface of the generated complex* **C**.

The complex-conics and cones present no peculiar features, *the con-*

* The singular surface of this complex is correctly described by Weiler in Art. 40 (B) of his Memoir (*loc. cit.*, p. 194). He errs, however, in ascribing to its congruence **G** the focal line $\overline{A_0B_0}$.

† Kummer, *loc. cit.*, Theorem xiv., p. 34.

gruence **G** *of proper singular lines, however, has now a focal surface which breaks up into two hyperboloids* S^2 *and* $\overline{S}^2$ *which have contact with each other along their common generator* $\overline{\alpha\beta}$. The congruence (4, 4) of common tangents to S^2 and $\overline{S}^2$ now breaks up, therefore, into the congruence **G**, and the linear congruence, twice counted, whose lines touch both hyperboloids at the several points of their common generator. To this last degenerate form, in fact, the congruence **Ġ** now reduces itself, whose lines are the proper singular lines of the complex **Ċ**, conjugate to **C** in the involution of Art. 27.*

33. The third case referred to in Art. 30 presents itself *whenever two conjugate lines* of the correlated planes, *as well as their corresponding points*, are brought into coincidence. In one and the same point C_0 now coincide, not only the points A_0, B_0, but also the self-conjugate points C_1, C_2, and with the latter, of course, the principal points A_1, A_2, B_1, B_2. Moreover, all the six principal lines a_0, a_1, a_2, b_0, b_1, b_2, are likewise coincident with one another, in $\alpha\beta$.

Every ordinary correlation now generates *a quadric* S^2 *which breaks up into a doubled plane* γ_0 *passing through* $\overline{\alpha\beta}$, *and bounded by a conic* s^2 *which touches* $\overline{\alpha\beta}$ *in* C_0. *This degenerate surface, together with the plane-pair* α, β, *supplemented by the doubled point* C_0, *is now the singular surface of the generated complex* **C**.

The pencil $(a_1a_2b_1b_2)$ includes, in addition to degenerate quadrics of the type just described, a series of cones, each of which, having contact with both planes α, β along their intersection $\overline{\alpha\beta}$, necessarily breaks up into a line-pair whose constituents are $\overline{\alpha\beta}$ and another line c_0, passing through C_0, whose position depends (as does that of γ_0) upon the correlation from which the degenerate quadric in question proceeds. The correlations which generate such cones are, however, exceptional ones, having a pair of associated singular points coincident in C_0.

The proper singular lines of every complex **C** *generated by an ordinary correlation, now form a congruence* **G** *which breaks up into a congruence* (1, 0), *consisting of the rays of the pencil* (C_0), *and an irreducible congruence* (1, 2) *which has* c_0 *for focal line, and the conic* s^2 *for focal curve.*†

34. In conclusion, it may be observed that the complexes of the second degree considered in the present note may, in virtue of the

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* The complex which forms the subject of Art. 32 is given by Weiler in Art. 20 of his Memoir. He desires me to state, however, that its complex-conics and cones are there erroneously described as having contact with the hyperboloid S^2 at points on its generator $\alpha\beta$.

† This complex is identical with the one briefly described by Weiler in Art. 45 (A) of his Memoir.

Principle of Duality, be also generated by two correlative pencils (A), (B) in space. Complex-lines are then the intersections of conjugate planes of the generating pencils. The singular surface of the complex breaks up into (1) the point-pair formed by the centres of the correlative pencils; (2) the two, real or imaginary, self-conjugate planes which these pencils always possess; and (3) the quadric surface which is the locus of the intersection of incident conjugate lines. The proper singular lines of the complex present themselves as the intersections of the planes which correspond to the several pairs of conjugate and incident lines in the two correlative pencils.

On a Theorem relating to Conformable Figures. *By* Prof. Cayley.

[*Read May 15th*, 1879.]

Consider two plane figures, say the figure of the points P referred to axes Ox, Oy, and that of the points P' referred to axes Ox', Oy'; and let x, y be the coordinates of P, and x', y' those of P'. If the figures correspond to each other in any manner whatever, P and P' being corresponding points, then we have x', y' each of them a function of x, y; and we may consider the second figure as derived from the first by altering the distance OP in the ratio $\sqrt{x'^2+y'^2} \div \sqrt{x^2+y^2}$, and by rotating it through the angle $\tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x}$; say by the Extension $\sqrt{x'^2+y'^2} \div \sqrt{x^2+y^2}$, and by the Rotation $\tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x}$; where the Extension and the Rotation are each of them a determinate function of x, y, the coordinates of P.

Passing from the point P to a consecutive point Q, the coordinates of which are $x+dx$, $y+dy$ (the ratio $dy \div dx$ being arbitrary), then the coordinates of the corresponding point Q' will be $x'+dx'$, $y'+dy'$, where $dx' = \frac{dx'}{dx}dx + \frac{dx'}{dy}dy$, $dy' = \frac{dy'}{dx}dx + \frac{dy'}{dy}dy$. Writing $\frac{dy'}{dx'}$ and $\frac{dy}{dx}$ instead of $dy' \div dx'$ and $dy \div dx$, the expressions $\sqrt{dx'^2+dy'^2} \div \sqrt{dx^2+dy^2}$ and $\tan^{-1}\frac{dy'}{dx'} - \tan^{-1}\frac{dy}{dx}$, will in general have values depending upon that of the arbitrary ratio $dy : dx$. But they may be independent of this ratio; viz., this is the case when x', y' are functions of x, y such that

$$\frac{dx'}{dy} = -\frac{dy'}{dx}, \quad \frac{dy'}{dy} = \frac{dx'}{dx};$$

and the two figures are then conformable (or conjugate) figures; that

is, figures similar as regards corresponding infinitesimal elements of area. We have, in this case, $\sqrt{dx'^2+dy'^2} \div \sqrt{dx^2+dy^2}$ and $\tan^{-1}\frac{dy'}{dx'} - \tan^{-1}\frac{dy}{dx}$ each a determinate function of x, y, the coordinates of P; and we pass from the element PQ to the corresponding element $P'Q'$ by altering the length in the ratio $\sqrt{dx'^2+dy'^2} \div \sqrt{dx^2+dy^2}$, and rotating the element through the angle $\tan^{-1}\frac{dy'}{dx'} - \tan^{-1}\frac{dy}{dx}$; say, this ratio and angle are the Auxesis and Streblosis respectively, these being, as already mentioned, functions of x, y only.

Considering now any two conformable figures, say the figure of the points P, and that of the points P'; we have the theorem that we can from the first figure obtain a third conformable figure by means of an Auxesis and a Streblosis which are respectively equal to the Extension and the Rotation by which the second figure is derived from the first.

In fact, if in the three figures respectively we take x, y, x', y', and x'', y'' for the coordinates of the corresponding points P, P', P'', the first and second figures are conformable, and we have therefore

$$\frac{dx'}{dy} = -\frac{dy'}{dx}, \quad \frac{dy'}{dy} = \frac{dx'}{dx};$$

the third figure is to have the Auxesis $\sqrt{x'^2+y'^2} \div \sqrt{x^2+y^2}$, and the Streblosis $\tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x}$; viz., writing r for $\sqrt{x^2+y^2}$, we ought to have

$$dx'' = \frac{xx'+yy'}{r^2}\,dx - \frac{xy'-x'y}{r^2}\,dy,$$

$$dy'' = \frac{xy'-x'y}{r^2}\,dx + \frac{xx'+yy'}{r^2}\,dy;$$

and it is therefore to be shown that there exist x'', y'' functions of x, y satisfying these relations; for, this being so, we have

$$\frac{dx''}{dy} = -\frac{dy''}{dx}, \quad \frac{dy''}{dy} = \frac{dx''}{dx};$$

and the third figure is thus conformable with the first.

Writing, for shortness,

$$A = \frac{xx'+yy'}{r^2}, \quad B = -\frac{xy'-x'y}{r^2},$$

the equations are

$$dx'' = Adx - Bdy,$$

$$dy'' = Bdx + Ady;$$

or the conditions for the existence of the functions x'', y'' are

$$\frac{dA}{dy} + \frac{dB}{dx} = 0, \quad \frac{dA}{dx} - \frac{dB}{dy} = 0.$$

We, in fact, have

$$\frac{dA}{dy}+\frac{dB}{dx}=\frac{1}{r^2}\left\{x\left(\frac{dx'}{dy}+\frac{dy'}{dx}\right)+y\left(\frac{dy'}{dy}-\frac{dx'}{dx}\right)+2y'\right\}$$
$$-\frac{2}{r^4}\{(xx'+yy')\,y+(xy'-x'y)\,x\}$$
$$=\frac{2y'}{r^2}-\frac{2}{r^4}(x^2+y^2)\,y'=0;$$

and similarly

$$\frac{dA}{dx}-\frac{dB}{dy}=\frac{1}{r^2}\left\{x\left(\frac{dx'}{dx}-\frac{dy'}{dy}\right)+y\left(\frac{dy'}{dx}+\frac{dx'}{dy}\right)+2x'\right\}$$
$$-\frac{2}{r^4}\{(xx'+yy')\,x-(xy'-x'y)\,y\}$$
$$=\frac{2x'}{r^2}-\frac{2}{r^4}(x^2+y^2)\,x'=0;$$

which proves the theorem.

The theorem is closely connected with the theory of the function of an imaginary variable; for, writing the conditions for the conformable figures in the form

$$\frac{dx'}{dx}=\frac{dy'}{dy}=F,\quad \frac{dx'}{dy}=-\frac{dy'}{dx}=-G,$$

we have

$$dx'=Fdx-Gdy,$$
$$dy'=Gdx-Fdy;$$

that is,

$$dx'+idy'=(F+iG)\,(dx+idy):$$

whence $F+iG$ is a function of $x+iy$, and then by integration $x'+iy'$ is also a function of $x+iy$. In one point of view any function such as $\phi(x,y)+i\psi(x,y)$ is a function of $x+iy$, for the quantity $x+iy$ is only known by means of its real components x, y; that is, knowing $x+iy$, we know x, y, and therefore also $\phi(x,y)+i\psi(x,y)$; and Cauchy, adopting this definition, introduced the expression "fonction monogène" of $x+iy$, to denote that which is in the more restricted (and the ordinary) sense termed a function of $x+iy$. And MM. Briot and Bouquet, in their "Théorie des fonctions elliptiques" (Paris, 1875), although not using Cauchy's expression *fonction monogène*, but the simple term *fonction*, do this under the qualification stated p. 3:—"Dans tout ce qui suit, nous ne nous occuperons que des fonctions qui admettent une dérivée." Now, a function admitting of a derivative (that is, in the ordinary sense, a function) of the imaginary variable z, $=x+iy$, is a function such that for a consecutive value z', $=x+iy+dx+idy$, we have

$$\frac{f(z')-f(z)}{z'-z}$$

= a quantity independent of the ratio of the real components dx, dy of the increment $dx+idy$ of the imaginary variable. Or, what is the

same thing, writing $f(z) = x' + iy'$, the condition in order that $x' + iy'$ may be a function of $x + iy$ is

$$dx' + idy' = (F + iG)(dx + idy),$$

where F and G are functions of x and y. It is not part of the condition that $F + iG$ shall be a function of $x + iy$, and it is only a long way further on that the authors prove that this is the case (see the definition of a "*fonction holomorphe*," p. 14; and the proof, p. 137). The last-mentioned equation $dx' + idy' = (F + iG)(dx + idy)$, where F and G are only assumed to be functions of x and y, has, if we represent $x + iy$ by means of the point P, coordinates (x, y), and in like manner $x' + iy'$ by means of the point P', coordinates (x', y'), the geometrical interpretation that the figures of the points P and P' are conformable figures, that is, figures similar as regards their infinitesimal elements. The foregoing theorem in regard to the Auxesis and Streblosis is that we can, by means of F and G, construct a third conformable figure,—in fact, the Auxesis and Streblosis are $= \sqrt{F^2 + G^2}$ and $\tan^{-1} \frac{G}{F}$ respectively; and, using these as an Extension and Rotation, we have the third conformable figure $x'' + iy'' = (F + iG)(x + iy)$; that is $(F + iG)(x + iy)$, and therefore also $F + iG$, is a function of $x + iy$,—and we have thus the derivative of a function of $x + iy$ as itself a function of $x + iy$.

It is to be remarked that, although the theorem of the Auxesis and Streblosis, considered as a property of conformable figures, is not by any means geometrically self-evident, yet the foregoing analytical proof is only a proof conducted by means of real quantities, of what (admitting the theory of imaginary quantities) is in fact self-evident; viz., the analytical conclusion really is that, F, G denoting functions of x, y, then, if $dx' + idy' = (F + iG)(dx + idy)$, that is, if $(F + iG)(dx + idy)$ be a complete differential, then $F + iG$ is a function of $x + iy$.

A Method of Constructing by Pure Analysis, functions X, Y, &c., which possess the property that $\int XY d\sigma = 0$, *and such that any given function can be expanded in the form* $\alpha X + \beta Y + \gamma Z + \ldots$ *By* E. J. Routh, F.R.S.

[*Read May 8th*, 1879.]

In many physical theories we have to expand some given function in a series of the kind indicated at the head of this paper. During the Christmas vacation, a method occurred to me of deducing by pure analysis, from one principle, all the expansions known to me. I was

unable to complete the method before the commencement of the following term, and was obliged to postpone the subject until the present time. I now venture to lay it before the Mathematical Society.

The following introduction contains a statement of certain propositions required in the theory. Two different applications of these immediately presented themselves according as (1) the expansion of the given function in terms of X, Y, &c., is to be true only for values of the independent variable restricted to lie between certain limits, or (2) to be always true. The functions X, Y, &c., are found to be any which satisfy a certain equation of an even order, and also certain other conditions which are not the same in the two applications. Two examples are given to illustrate these results. Finally, the theory is extended to the case of two independent variables, and illustrated by an example taken from Laplace's Functions.

I hope on another occasion to lay a further communication before the Society.

Introduction.

1. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the coordinates of the extremities of three conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Construct the determinant

$$\begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{b} & \frac{z_1}{c} \\ \frac{x_2}{a} & \frac{y_2}{b} & \frac{z_2}{c} \\ \frac{x_3}{a} & \frac{y_3}{b} & \frac{z_3}{c} \end{vmatrix}$$

Then (i.) the sum of the products two and two of the constituents in any two horizontal lines is zero, but if the horizontal lines be the same the sum is unity.

(ii.) The sum of the products two and two of the constituents in any two vertical lines is zero, except the two vertical lines are the same, and then the sum is unity.

(iii.) Each constituent is equal to its minor, if the proper extremities of the conjugate diameters are chosen; and the whole determinant is equal to unity.

All these results follow at once from the consideration that the constituents of the determinant are the direction-cosines of three straight lines at right angles.

2. The chief object of the following paper is to show, that the first of the properties just mentioned corresponds to that property of Laplace's Functions, and certain other functions, which is usually written

$$\int Y_m Y_n d\sigma = 0,$$

where m and n are not equal; that the second property corresponds to the proposition that any function can be expanded in a series of the form

$$a_0 Y_0 + a_1 Y_1 + a_2 Y_2 + \ldots$$

The third property may be conveniently used to show that the expansion can be made in one way only.

3. In order to establish this correspondence, it will be necessary to extend our propositions to quadrics of any number of variables. Let then

$$A_1^2 X_1^2 + A_2^2 X_2^2 + A_3^2 X_3^2 + \ldots + A_n^2 X_n^2 = H^2$$

be any quadric function of the n variables $X_1, X_2, \ldots X_n$; A_1, A_2, &c., and H being any constants.

Let us call $(X_1 \ldots X_n)$, $(Y_1 \ldots Y_n)$, &c. the n coordinates of the n extremities of any set of conjugate diameters. That is, let these points be such that the diametral plane of each contains all the others. Construct now the determinant

$$\begin{vmatrix} A_1 X_1 & A_2 X_2 & A_3 X_3 & \&c. \\ A_1 Y_1 & A_2 Y_2 & A_3 Y_3 & \&c. \\ A_1 Z_1 & A_2 Z_2 & A_3 Z_3 & \&c. \\ \&c. & \&c. & \&c. & \end{vmatrix}$$

Then the first of our preliminary propositions may be written analytically in the forms

$$A_1^2 X_1 Y_1 + A_2^2 X_2 Y_2 + \ldots = 0,$$
$$A_1^2 X_1 Z_1 + A_2^2 X_2 Z_2 + \ldots = 0,$$
$$\&c. = 0,$$

except when $X = Y$ or $X = Z$, &c., and then the sum is equal to H^2.

The second preliminary proposition may be written in the forms

$$A_1 A_2 (X_1 X_2 + Y_1 Y_2 + \ldots) = 0,$$
$$A_1 A_3 (X_1 X_3 + Y_1 Y_3 + \ldots) = 0,$$
$$\&c. = 0,$$

except when the suffixes are the same, and then the sum is equal to H^2.

The third preliminary proposition may be written in the same form as before.

4. The second of these properties really follows from the first, so that, whenever the first is true for any determinant, the second will be true also. To prove this, let us put

$$\begin{aligned} A_1X_1\xi + A_1Y_1\eta + \ldots &= xH, \\ A_2X_2\xi + A_2Y_2\eta + \ldots &= yH, \\ \&c. &= \&c. \end{aligned}$$

Then the first property is equivalent to the statement

$$\xi^2 + \eta^2 + \ldots = x^2 + y^2 + z^2 + \ldots$$

for all values of ξ, η, &c. But if we multiply the equations by A_1X_1, A_2X_2, &c., and add the results, we get

$$H\xi = A_1X_1x + A_2X_2y + \ldots;$$

similarly

$$H\eta = A_1Y_1x + A_2Y_2y + \ldots$$

Adding the squares of these equations, and equating the coefficients of the independent variables x, y, &c., we get the second property.

This proposition will show how closely the two properties put into the heading of this paper are connected with each other.

5. There is one other proposition relating to conjugate diameters which will be wanted. When the given quadric is not referred to its principal coordinates, the second property given above is true, but with some modification. We then have

$$\begin{aligned} X_1X_2 + Y_1Y_2 + \ldots &= \text{constant}, \\ X_1^2 + Y_1^2 + \ldots &= \text{constant}, \\ \&c. \qquad \&c. & \end{aligned}$$

for all sets of conjugate diameters of the same quadric.

6. We have now to select one complete set of conjugate diameters. To do this we assume any second quadric we please. Let this be

$$U = \tfrac{1}{2}a_{11}X_1^2 + a_{12}X_1X_2 + \ldots = 1,$$

so that the constants a_{11}, a_{12}, &c. may be chosen at our pleasure. We shall then find the set of conjugate diameters common to this second quadric and the first quadric.

Let X_1, $X_2 \ldots X_n$ be the coordinates of the extremity of one such diameter in the first quadric. Then we have

$$\left.\begin{aligned} \frac{dU}{dX_1} &= pA_1^2X_1 \\ \frac{dU}{dX_2} &= pA_2^2X_2 \\ \&c. &= \&c. \end{aligned}\right\}$$

where p is to be found from the determinantal equation

$$\begin{vmatrix} a_{11}-pA_1^2 & a_{12} & \&c. \\ a_{12} & a_{22}-pA_2^2 & \&c. \\ & \&c. & \end{vmatrix} = 0.$$

We thus have n values of p. Each value of p determines the n coordinates of the extremity of one of the conjugate diameters.

If we divide the constituents of the first row and the first column by A_1, those of the second row and column by A_2, and so on, we reduce this determinant to the form given in Art. 44 of Dr. Salmon's "Higher Algebra." The roots of this determinantal equation are therefore all real. The ratios of X_1, X_2, &c. are therefore also all real.

7. The case in which some of the roots are equal will be found to be important in what follows. It may be shown* that such equalities imply a corresponding indeterminateness in the values of the ratios of $X_1, X_2 \ldots X_n$. If one of these ratios is arbitrary, two of the roots will be found equal; if two ratios are arbitrary, three of the roots will be equal, and so on.

8. In beginning with these properties of conjugate diameters, we are not really resting the propositions which follow on a geometrical basis.

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* Let the determinantal equation be written $f(p) = 0$. It is proved in Dr. Salmon's "Sixth Lesson on Higher Algebra," that if a new determinantal equation be found by equating to zero the minor of any constituent in the leading diagonal, the roots of *this* equation separate the roots of $f(p)$. If, then, two of the roots of $f(p)$ are equal, it follows that all the minors of every constituent in the leading diagonal are zero. We can now prove that the minor of every constituent is zero. For, consider the constituent a_{12}. Since the determinant vanishes, the square of its minor is (by Ex. 1, Art. 32 of Salmon's Algebra) = (minor of $a_{11}-pA_1^2$) (minor of $a_{22}-pA_2^2$), which is zero. But any other constituent can be brought into the place of a_{12} by changes which keep the determinant symmetrical. Hence the minor of every constituent is zero. But, these are just the conditions that all the equations in Art. 3 can be satisfied in an infinite number of ways by the ratios $X_1 : X_2$: &c. If there be three equal roots, we can show that all the second minors are zero, and that two ratios are indeterminate; see also pages 6 and 7 of the Author's Treatise on the "Stability of Motion."

Conversely, let there be an indeterminateness in the equations, so that one ratio is arbitrary. The analytical condition is that all the first minors are zero. Let us call these minors B_{11}, B_{12}, &c. Then

$$\frac{df(p)}{dp} = -B_{11}A_1^2 - B_{22}A_2^2 - \ldots$$

If, therefore, all the minors of the constituents in the leading diagonal be zero, this differential coefficient is zero, and the determinant has two equal roots. If there be an indeterminateness so that two ratios are arbitrary, all the second minors are zero. But we have $\frac{d^2f(p)}{dp^2} = -A_1^2\frac{dB_{11}}{dp} - A_2^2\frac{dB_{22}}{dp} - \&c.$

Hence this differential coefficient vanishes, and the equation has two equal roots.

We merely use the known results of geometry to shorten the demonstrations. We might, if we pleased, give purely analytical proofs of these properties without a word of allusion to any geometrical construction.

First Application.

Case in which limits are given.

9. The quadric U is perfectly arbitrary, let us take

$$\begin{aligned} 2U = \quad & a_1 X_1^2 + b_1 (X_2 - X_1)^2 \\ & + a_2 X_2^2 + b_2 (X_3 - X_2)^2 \\ & + \&c. \\ & + a_{n-1} X_{n-1}^2 + b_{n-1} (X_n - X_{n-1})^2 \\ & + a_n X_n^2. \end{aligned}$$

Then, taking any equation, say the x^{th},

$$\frac{dU}{dX_x} = pA_x^2 X_x,$$

we have $\quad a_x X_x + b_{x-1}(X_x - X_{x-1}) - b_x (X_{x+1} - X_x) = pA_x^2 X_x.$

Suppose we regard X_x as a function of x and p, so that, when $x = h$, $X = X_1$, when $x = 2h$, $X = X_2$, and so on. Then, this equation is an equation of Differences, which may be also written in the form

$$a_x X_x - \Delta (b_{x-1} \Delta X_{x-1}) = pA_x^2 X_x.$$

If we regard h as indefinitely small, and write b' for $h^2 b$, the equation takes the form $\quad a_x X - \dfrac{d}{dx}\left(b'_x \dfrac{dX}{dx}\right) = pA_x^2 X,$

which is a Differential equation of the second order.

10. But neither of these two general forms includes the first or the last equation. In order that the equation corresponding to $x = h$ may be included, we must suppose it to be derived by the law of the general term from the quadric

$$U + \tfrac{1}{2}\lambda X_1^2,$$

where λ is some quantity which is independent of p. That is, we must have $\quad a_1 X_1 + b_0 (X_1 - X_0) - b_1 (X_2 - X_1) = pA_1^2 X_1$

the same as $\quad a_1 X_1 - b_1 (X_2 - X_1) + \lambda X_1 = pA_1^2 X_1.$

This requires $\quad b_0 (X_1 - X_0) = \lambda X_1,$

or in the limit when h is indefinitely small

$$b'_0 \frac{dX}{dx} = \lambda' X,$$

where $\lambda' = \lambda h$. This equation holds when $x = 0$.

In just the same way the last equation corresponding to $x = nh$ will be included in the general form if we have

$$-b_n (X_{n+1} - X_n) = \mu X_n.$$

In the limit when h is indefinitely small, this becomes

$$-b'_n \frac{dX}{dx} = \mu' X,$$

where μ' has been written for μh. This equation holds when $x = nh$, which in the limit when n is infinite we shall write $x = l$.

11. For the sake of brevity, we shall often speak only of the Differential equation, though the remarks apply also, with slight modifications, to the equation of Differences.

Let then a function X be given, between the limits $x = 0$ and $x = l$, by the differential equation

$$a_x X - \frac{d}{dx}\left(b'_x \frac{dX}{dx}\right) = p A_x^2 X,$$

where a_x, b'_x, and A_x are given functions of x, but are independent of p. Also, let it be given that, when $x = 0$,

$$b'_0 \frac{dX}{dx} = \lambda' X;$$

and when $x = l$, $$-b'_l \frac{dX}{dx} = \mu' X,$$

where λ' and μ' are independent of p.

If we solve the differential equation, we have

$$X = L\psi_1 (x, p) + M\psi_2 (x, p),$$

where L and M are undetermined constants. One of the two conditions at the limits gives the ratio of L to M. We thus have

$$X = P\phi (x, p),$$

where P is a function of p only. The condition at the other limit will now give an equation to determine p. If p, q, &c. be the values of p thus found, we may write

$$Y = Q\phi (x, q),$$
$$Z = R\phi (x, r),$$
$$\&c.$$

12. This equation, to find p, must be the same as the determinantal equation given in Art. 6, though in an expanded form. For, though no mention has been made of that equation, yet it was deduced from the n equations to find $X_1 \ldots X_n$, and is therefore virtually included in

those equations. These are the very equations from which we now deduce an equation to find p.*

13. It appears, from Art. 6, that the values of p, q, &c., thus found, are all real. Since there is no indeterminateness in the value of X, it also appears, from Art. 7, that no two of the roots are equal. In the case of the equation of Differences, the number of roots is exactly n. In the Differential equation, the number is infinite.

14. The values of P, Q, &c. may be easily found by substituting in the equation

$$A_1^2 X_1^2 + A_2^2 X_2^2 + \ldots = H^2.$$

Thus, writing $Hh = H'$, we have

$$P^2 \int_0^2 \{\phi(x, p)\}^2 dx = H'^2,$$

with similar expressions for Q, R, &c.

15. It also appears, from Art. 3, that in the case of the equation of Differences, we have

$$\Sigma XYA_x^2 = 0,$$
$$\Sigma XZA_x^2 = 0,$$
$$\&c. = 0,$$

the summations being taken from $x = 0$ to $x = nh$.

In the case of the Differential equation, we have

$$\int_0^2 XYA_x^2 dx = 0,$$
$$\int_0^2 XZA_x^2 dx = 0,$$
$$\&c. = 0.$$

16. The quantities λ, μ are constants at our disposal, so that, by giving them different values, we may sometimes obtain from the same differential equation more than one set of functions. Each of these sets will possess the two properties mentioned in the heading to this paper. These properties are also enunciated in Art. 15 and Art. 17.

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* In the case of the equation of Differences we get, by continual substitution, beginning at the first equation,

$$X_n = X_0 \psi_1(x, p) + X_1 \psi_2(x, p),$$

where X_0 and X_1 are undetermined, and ψ_1, ψ_2 are functions of p of the $n-1^{\text{th}}$ degree. Introducing the given relation between X_0 and X_1, we have

$$X_n = P\phi(x, p),$$

where ϕ is a function of p of the $n-1^{\text{th}}$ degree. Substituting, now, in the given relation between X_n and X_{n+1}, the factor P divides out, and we have left an equation of the n^{th} degree to find p. Thus the two methods of obtaining this equation are substantially the same, and lead to an equation of the same degree.

17. Let us next inquire whether any function, say $f(x)$, can be expanded in a series of the form

$$f(x) = a\phi(x, p) + \beta\phi(x, q) + \ldots$$

Let the values of $f(x)$, when $x = h$, $2h$, $3h \ldots nh$, be B_1, B_2, $B_3 \ldots B_n$. These values we assume to be all finite. Consider now the series

$$\begin{aligned} S = \quad & X_x (B_1 A_1^2 X_1 + B_2 A_2^2 X_2 + \ldots) \\ & + Y_x (B_1 A_1^2 Y_1 + B_2 A_2^2 Y_2 + \ldots) \\ & + \&c. \end{aligned}$$

When $x = h$, we have $X_x = X_1$, $Y_x = Y_1$, &c.; and therefore, by Art. 3, $S = B_1 H^2$. Again, when $x = 2h$, we have $X_x = X_2$, $Y_x = Y_2$, &c.; and therefore $S = B_2 H^2$, and so on for the other values of x.

Substituting for X_x, Y_x, &c. their values given by Art. 11, we see that the expansion

$$f(x) = a\phi(x, p) + \beta\phi(x, q) + \ldots$$

has n terms, and is true for the values $x = h$, $x = 2h$, &c. up to $x = nh$, provided

$$a = \frac{\Sigma\phi(x, p) A_x^2 f(x)}{\Sigma [\phi(x, p) A_x]^2}, \quad \beta = \frac{\Sigma\phi(x, q) A_x^2 f(x)}{\Sigma [\phi(x, q) A_x]^2}, \text{ &c.}$$

The summations in the values of a, β, &c. extend from $x = h$ to $x = nh$.

18. In the limit, when h is indefinitely diminished, the expansion

$$f(x) = a\phi(x, p) + \beta\phi(x, q) + \ldots$$

has an infinite number of terms; but the number is equal to the number of roots in the equation to determine p. In this case the equation is true for all values of x between $x = 0$ and $x = l$, and the values of a, β, &c. are

$$a = \frac{\int_0^l \phi(x, p) A_x^2 f(x)\, dx}{\int_0^l [\phi(x, p) A_x]^2\, dx}, \quad \beta = \frac{\int_0^l \phi(x, q) A_x^2 f(x)\, dx}{\int_0^l [\phi(x, q) A_x]^2\, dx}, \text{ &c.}$$

19. It may be noticed that no assumption has been made as to the continuity of the function $f(x)$. It has, however, been assumed to be finite between the limits.

20. There can be but one expansion of $f(x)$ in a series of functions of the form $\phi(x, p)$; for, if there were another, let it be

$$f(x) = a'\phi(x, p) + \beta'\phi(x, q) + \ldots$$

We have, by subtraction,

$$0 = a\phi(x, p) + b\phi(x, q) + \dots,$$

where $a = \alpha - \alpha'$, $b = \beta - \beta'$, &c. Writing for x its values h, $2h \dots nh$, we have n equations to find the n constants a, b, c, &c. These are

$$0 = aX_1 + bY_1 + \dots,$$
$$0 = aX_2 + bY_2 + \dots,$$
$$0 = \&c.$$

Unless a, b, &c. are all zero, we have their ratios equal to the ratios of the minors of the constituents of any line, say the first line. But these minors, by the third preliminary proposition, are respectively equal to X_1, Y_1, &c. Substituting in the first line, we have

$$0 = X_1^2 + Y_1^2 + \dots,$$

which contradicts the third preliminary proposition.

21. None of the values of $X_1 X_2 \dots X_n$ can be infinite, since they satisfy the quadric

$$A_1^2 X_1^2 + A_2^2 X_2^2 + \dots + A_n^2 X_n^2 = H^2,$$

provided $A_1^2 A_2^2 \dots A_n^2$ all keep one sign. It immediately follows that the function $\phi(x, p)$ cannot be infinite for any value of x between the given limits. For P the multiplier is independent of x, and therefore $P\phi(x, p) = X$ could not be finite for all values of x. This argument applies whether the terminal conditions are identically satisfied or not. So that, if either ψ_1 or ψ_2 in

$$X = L\psi_1(x, p) + M\psi_2(x, p)$$

becomes infinite between the limits, then the coefficients L or M, or both, must be zero.

22. We may also notice that, if X_1 is finite and b_1 not zero, the first equation in Art. 9 requires X_2 to be finite; and, granting this, we have from the general equation in the same article

$$b_x X_{x+1} = (b_{x-1} + b_x + a_x - pA_x^2) X_x - b_{x-1} X_{x-1}.$$

If X_x, X_{x-1} are finite, and b_x not zero between the limits, then X_{x+1} is finite. Thus, if the terminal conditions exist, no infinite values of X can enter. Thus both integrals $\psi_1(x, p)$ and $\psi_2(x, p)$ will be finite throughout if they are finite when $x = 0$ and b_x keeps one sign.

We also notice that, when $X_x = 0$, X_{x-1} and X_{x+1} have opposite signs; hence X_x changes sign whenever it vanishes. It easily follows also from the equation that $X_x = 0$ has no equal roots.

23. *The functions which possess the property*

$$\int_0^2 \phi(x, p)\, \phi(x, q)\, A_x dx = 0$$

are not restricted to be the solutions of a differential equation of the second order. We may take a more general value for U, say

$$\begin{aligned} 2U = {} & a_1 X_1^2 + b_1 (X_2 - X_1)^2 + c_1 (X_3 - 2X_2 + X_1)^2 \\ & + e_1 (X_4 - 3X_3 + 3X_2 - X_1)^2 \\ & + \text{four similar terms beginning with } X_2, \\ & + \&c., \text{ beginning with } X_3,\ X_4,\ \&c. \end{aligned}$$

Then, taking any equation, say the x^{th}, we have

$$\frac{dU}{dX_x} = pA_x^2 X_x.$$

For the sake of brevity, we shall write down only the terms which contain c on the left-hand side. These are

$$\begin{aligned} & c_{x-2} (X_x - 2X_{x-1} + X_{x-2}) \\ & \quad - 2c_{x-1} (X_{x+1} - 2X_x + X_{x-1}) \\ & \quad + c_x (X_{x+2} - 2X_{x+1} + X_x). \end{aligned}$$

These may be written in the usual notation

$$\Delta^2 (c_{x-2} \Delta^2 X_{x-2}).$$

In the limit when h is indefinitely small, we have

$$h^4 \frac{d^2}{dx^2} \left(c \frac{d^2 X}{dx^2} \right).$$

The result is that, if we form the equation of Differences

$$a_x X_x - \Delta (b_{x-1} \Delta X_{x-1}) + \Delta^2 (c_{x-2} \Delta^2 X_{x-2}) - \&c. = pA_x X_x,$$

or the Differential equation

$$a_x X - \frac{d}{dx} \left(b_x \frac{dX}{dx} \right) + \frac{d^2}{dx^2} \left(c_x \frac{d^2 X}{dx^2} \right) - \&c. = pA_x X;$$

then, in the first case, we have

$$\Sigma_1^n XYA_x^2 = 0,$$

the summation extending from $x = h$ to $x = nh$; and, in the second case,

$$\int_0^l XYA_x^2 dx = 0.$$

24. *We have still to consider the equations at each end.* We observe that the terms with zero and negative suffixes are missing. But, exactly as in Art. 10, we may add on to the expression for $2U$ the terms

$$\lambda X_1^2 + \mu (X_2 - X_1)^2 + \nu (X_3 - 2X_2 + X_1)^2.$$

To find the conditions at the limits, we equate the missing terms to

the differential coefficients of these added terms with regard to X_1, X_2, and X_3.

It seems unnecessary to repeat the argument. After some reduction we obtain the conditions

$$\begin{aligned} b_0 \Delta X_0 - \Delta (c_{-1} \Delta^2 X_{-1}) + \&c. &= \lambda X_1, \\ c_0 \Delta^2 X_0 - \Delta (e_{-1} \Delta^3 X_{-1}) + \&c. &= \mu \Delta X_1, \\ \&c. &= \&c. \end{aligned}$$

for the equation of Differences; and

$$\begin{aligned} b'_x \frac{dX}{dx} - \frac{d}{dx}\left(c'_x \frac{d^2X}{dx^2}\right) + \&c. &= \lambda' X, \\ c'_x \frac{d^2X}{dx^2} - \frac{d}{dx}\left(e'_x \frac{d^3X}{dx^3}\right) + \&c. &= \mu' \frac{dX}{dx}, \\ \&c. &= \&c. \end{aligned}$$

for the Differential equation.

We have similar conditions at the other end, the constants λ', μ', &c. not being necessarily the same at both extremities.

25. If the differential equation be of the $2n^{\text{th}}$ order, we have thus n conditions at each end. All the constants of integration except one will therefore be determined. There will also be an equation to determine p, the roots of which are all real and unequal.

26. Exactly as in Art. 17, it follows that any function $f(x)$ can be expanded in a series of functions of the form

$$f(x) = \alpha\phi(x, p) + \beta\phi(x, q) + \ldots,$$

where $X = P\phi(x, p)$ is the solution of the differential equation which satisfies the conditions at the limits, and is finite throughout.

Second Application.

Case in which no limits are given.

27. We may take another form for the quadric U which leads to another result. Suppose we have

$$\begin{aligned} 2U = {} & a_1 X_1^2 + b_1 (X_2 - X_1)^2 \\ & + \&c. \\ & + a_{n-1} X_{n-1}^2 + b_{n-1} (X_n - X_{n-1})^2 \\ & + a_n X_n^2 + b_1 (X_1 - X_n)^2, \end{aligned}$$

which differs from that taken in Art. 9 by the addition of the last term. Let us also suppose that a_x, b_x, A_x, and X are functions which repeat so that $a_{n+1} = a_1$, $a_{n+2} = a_2$, and so on.

It is now evident that all the previous reasoning applies as before except that we have no conditions at the limits.

28. Thus, supposing we have the differential equation

$$a_x X - \Delta(b_x \Delta X) = p A_x X;$$

when solved, we have

$$X = L\phi(x, p) + M\psi(x, p).$$

But if X is to repeat, we have

$$X_1 = X_{n+1}, \quad X_2 = X_{n+2}.$$

Since the equation of Differences gives X_3 in terms of X_1 and X_2, these are the necessary and sufficient conditions of repetition. In the case of a Differential equation, we should have $X_0 = X_n$, $\frac{dX_0}{dx} = \frac{dX_n}{dx}$. These equations determine the ratio of L to M, and also give an equation to find the values of p.

29. In the case considered in Art. 9, the equations to find X_1, X_2, &c. are such that each contains a letter which is in none of the other equations. In fact, if b_x does not vanish between the limits, the equations are all independent, and X_1, X_2, &c. are all determinate. We therefore infer, by Art. 7, that the equation to find p has no equal roots. But this argument does not apply to the quadric now under consideration. It is therefore possible that the two equations given in the last article may really leave X indeterminate with two arbitrary constants.

30. In this case, the method described in Art. 6 fails to separate one particular set of conjugate diameters. But a simple consideration will enable us to complete the process. For example, let there be two roots equal to p. Then we have

$$\left.\begin{aligned} X &= L\phi(x, p) + M\psi(x, p) \\ Y &= L'\phi(x, p) + M'\psi(x, p) \end{aligned}\right\}.$$

The extremity of one diameter is arbitrary, so that the ratio L to M may be what we please. But, when this has been chosen, the ratio of L' to M' must be such that

$$\Sigma A_x^2 XY = 0.$$

This gives

$$LL'\Sigma(A^2\phi^2) + (LM' + L'M)\,\Sigma(A^2\phi\psi) + MM'\Sigma(A^2\psi^2) = 0.$$

Also, since the extremities of the diameters must lie on the quadric, we have, as in Art. 14,

$$\begin{aligned} L^2\,\Sigma(A^2\phi^2) + 2LM\ \Sigma(A^2\phi\psi) + M^2\,\Sigma(A^2\psi^2) &= H^2, \\ L'^2\,\Sigma(A^2\phi^2) + 2L'M'\,\Sigma(A^2\phi\psi) + M'^2\,\Sigma(A^2\psi^2) &= H^2, \end{aligned}$$

where H is practically a constant at our disposal.

31. It might be supposed that, having an undetermined constant L at our disposal, we might, by following the rule in Art. 17, obtain more than one expansion of any function in a series of the form

$$f(x) = \alpha[L\phi x + M\psi x] + \beta[L'\phi x + M'\psi x] + \&c.$$

But this is not the case, so that in general it is more convenient to give L any value we please.

To prove this we follow the same rule as in Art. 17, and obtain

$$\alpha = \frac{L\Sigma\phi f A_x^2 + M\Sigma\psi f A_x^2}{H^2}, \quad \beta = \frac{L'\Sigma\phi f A_x^2 + M'\Sigma\psi f A_x^2}{H^2}.$$

Hence, the coefficient of $\phi(x)$ in the expansion of $f(x)$ is

$$\frac{L^2+L'^2}{H}\,\Sigma\phi f A_x^2 + \frac{LM+L'M'}{H}\,\Sigma\psi f A_x^2.$$

Now, looking at the equations given in the last article, we see that $(L, M)(L', M')$ may be regarded as the coordinates of two points lying on the quadric

$$L^2\Sigma(A^2\phi^2) + 2LM\Sigma(A^2\phi\psi) + M^2\Sigma(A^2\psi^2) = H,$$

while the condition

$$LL'\Sigma(A^2\phi^2) + (LM'+L'M)\,\Sigma(A^2\phi\psi) + MM'\Sigma(A^2\psi^2) = H$$

expresses the fact that these points are the extremities of conjugate diameters. Hence, by Art. 5, $L^2+L'^2$, $LM+L'M'$, and $M^2+M'^2$ are the same for all values of L.

The case of several equal roots may be treated in the same way.

Example 1.

32. *Under what circumstances can we expand any finite function $f(x)$ of x in a series of functions which satisfy*

$$\frac{d^2X}{dx^2} + pX = 0?$$

We have, in this case,

$$X = L\sin\sqrt{p}\,x + M\cos\sqrt{p}\,x.$$

If the expansion is to be true for all values of x, say between $x=0$ and $x=2\pi$, and not to be necessarily true for any values of x outside these limits, we must have a relation

$$\frac{dX}{dx} = \lambda X$$

true at each end. If we take $\lambda = 0$, we have

$$L\cos\sqrt{p}\,x - M\sin\sqrt{p}\,x = 0,$$

both where $x=0$ and $x=2\pi$. This excludes negative values of p, and gives $L=0$, $\surd p=\frac{1}{2}i$, where i is any integer. Thus we have

$$f(x)=\Sigma M\cos\frac{ix}{2},$$

where the summation extends to all integer values of i, and M can be found in the usual manner. If we take λ infinite, we obtain an expansion in sines alone. Any other value of λ will give an expansion in both sines and cosines.

It is important to notice that a series in sines or cosines alone does not necessarily repeat when x is increased by 2π.

33. If the function $f(x)$, whether continuous or not, repeats when x is increased by 2π, and the expansion is to be true for *all* values of x, then we must use the method of Art. 27. In this case p must be a square integer, and there is nothing to determine the ratio L to M. This indeterminateness, by Art. 7, shows that the roots $p=1^2$, $p=2^2$, &c. occur twice each time, but the root $p=0$ only once. We thus have

$$\begin{aligned}X&=L_0,\\ Y&=L_1\sin x+M_1\cos x,\\ Z&=L_1'\sin x+M_1'\cos x,\end{aligned}$$

with similar terms for the other values of p. We have also, by Art. 30, the conditions

$$L_1L_1'+M_1M_1'=0,$$

with similar equations corresponding to the other values of p. These are satisfied by $L_1'=0$, $M_1=0$. Hence the expansion

$$f(x)=\alpha X+\beta Y+\gamma Z+\ldots$$

of Art. 31 becomes

$$f(x)=\alpha+\beta\sin x+\gamma\cos x+\delta\sin 2x+\epsilon\cos 2x+\ldots,$$

where the coefficients are given by Art. 17, or may be found in the usual way.

Example 2.

34. *Under what circumstances can we expand any finite function $f(x)$ of x in a series of functions which satisfy*

$$\frac{d}{dx}\left\{(1-x^2)\frac{dX}{dx}\right\}+pX=0,$$

between the limits $x=-1$ to $x=l$.

First we notice that this equation comes under the general form given in Art. 9, where $b=1-x^2$. But this case is peculiar, because at the limit $x=-1$, $b=0$, so that the condition

$$b\frac{dX}{dx}=\lambda X,$$

when λ is chosen to be zero, appears to be nugatory. We may, however, conclude that, if X be any solution of the equation, and Y be the value of X when q is written for p, and if $\frac{dX}{dx}$ and $\frac{dY}{dx}$ are finite at both limits, then

$$\int_{-1}^{+1} XY\,dx = 0,$$

if p and q be unequal.

35. In order, however, to expand $f(x)$ in a series of the functions X, Y, &c., it will be necessary to find *all* the values of p which satisfy the limiting conditions.

To examine the condition at the lower limit, let us put $x = -1+t$, and expand X in powers of t. We have

$$\frac{d}{dt}\left\{(2t-t^2)\frac{dX}{dt}\right\} + pX = 0.$$

To find one solution, we put

$$X = A_0 + A_1 t + A_2 t^2 + \ldots$$

We have, by substitution,

$$A_{n+1} = \frac{n(n+1)-p}{2(n+1)^2} A_n.$$

By taking the test fraction, we find the series is convergent up to $t=2$.

Let the solution thus found be $X = f(t)$. Then the general solution is

$$X = Lf(t)\int \frac{dt}{b\{f(t)\}^2} + Mf(t),$$

where $b = 2t-t^2$. We easily find, when $t=0$, that

$$b\frac{dX}{dt} = \frac{L}{A_0}, \qquad X = \frac{L}{2A_0}\log t + MA_0.$$

The constant λ is at our disposal, except that by Art. 10 it must be independent of p. If λ be finite, it does not appear to be possible to satisfy the condition

$$b\frac{dX}{dt} = \lambda X,$$

when $t=0$ with the value of X just found unless $L=0$. If we choose $\lambda = 0$, the condition at the lower limit becomes

$$b\frac{dX}{dt} = 0,$$

and this requires $L=0$. If we choose $\lambda = \infty$, the condition becomes $X=0$, which requires $L=0$ and $M=0$. The function X is then always zero.

To satisfy the condition at the lower limit, we must therefore take for X the series represented above by $f(t)$.

36. If l be less than unity, we have p determined by the condition

$$(1-x^2)\frac{dX}{dx} = \mu X,$$

which holds at the upper limit. But if $l=1$, we have $b=1-x^2=0$.

By the same reasoning as before, it does not appear to be possible to satisfy this condition unless

$$b\frac{dX}{dx} = 0$$

at the upper limit, and, as before, we get the same results as if we put $\mu=0$.

If we choose $\mu=0$, this condition is satisfied if p be any integer of the form $m(m+1)$. At the same time, the condition may be satisfied by other values of p, and we must therefore examine the value of the left-hand side of this equation when $t=2$. Now

$$\begin{aligned}(2-t)\frac{dX}{dt} &= (2-t)(A_1+2A_2t+\ldots)\\ &= \text{zero}+(2-t)\{rA_rt^{r-1}+(r+1)A_{r+1}t^r+\ldots\}\\ &= t^{r-1}[2rA_r+\{2(r+1)A_{r+1}-rA_r\}t+\ldots]\\ &= t^{r-1}p\left\{\frac{2r}{p}A_r-\frac{A_rt}{r+1}-\frac{A_{r+1}}{r+2}t^2-\ldots\right\},\end{aligned}$$

which is obtained by substituting for A_{r+1} its value in terms of A_r, and so on. Now r is any integer we please; let it be greater than p. Then every term after the first in this series has the same sign; and since

$$\frac{A_{n+1}}{n+2} > \frac{A_n}{n+1}\cdot\frac{n^2}{(n+1)(n+2)}\cdot\frac{1}{2},$$

the sum, when $t=2$, of all the terms after the first is greater than

$$\frac{A_r}{r+1}\left\{1+\frac{r^2}{(r+1)(r+2)}+\ldots\right\},$$

which is a divergent series. We, therefore, conclude that we cannot have $(1-x^2)\frac{dX}{dt}=0$ unless p is an integer of the form $m(m+1)$.

37. *The result is, that if we wish to expand $f(x)$ in a series of functions which satisfy the given equation, and if the expansion is to hold for all values of x between ±1, then the several terms of the expansion are given by $p=0$, $p=1.2$, $p=2.3$, &c.; and the general term by $p=m(m+1)$.*

THIRD APPLICATION.

Case of two Independent Variables.

38. Let us now apply the method to the case of two independent variables, x and y. Suppose we take the quadric

$$2U = a_{11}X_{11}^2 + b_{21}(X_{21}-X_{11})^2 + c_{12}(X_{12}-X_{11})^2 + e_{11}(X_{21}-X_{11})(X_{12}-X_{11})$$
$$+ \text{similar terms with continued additions of unity to each suffix.}$$

Here we take with X_{11}^2 only the products of X_{11} with X_{12} and X_{21} and the product $X_{12}X_{21}$.

Taking any one of the equations in Art. 9, say

$$\frac{dU}{dX_{xy}} = pA_{xy}^2 X_{xy},$$

we find, by the same process as before, the equation of Differences

$$a_{xy}X_{xy} - \Delta_x(b_{x-1,y}\Delta_y X_{x-1,y}) - \Delta_y(c_{x,y-1}\Delta_y X_{x,y-1})$$
$$- \Delta_x(e_{x-1,y}\Delta_y X_{x-1,y}) - \Delta_y(e_{x,y-1}\Delta_x X_{x,y-1}) = pA_{xy}^2 X_{xy}.$$

In the limit this becomes the differential equation

$$\left.\begin{aligned} aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) - \frac{d}{dy}\left(c\frac{dX}{dy}\right) \\ - \frac{d}{dx}\left(e\frac{dX}{dy}\right) - \frac{d}{dy}\left(e\frac{dX}{dx}\right) \end{aligned}\right\} = pA^2X,$$

where a, b, c, e and A are any functions of x, y, but must be independent of p.

39. The condition at the limits is as follows:—If s be the arc of the boundary of the area over which we integrate, then at every point of the boundary

$$b\frac{dX}{dx}\frac{dy}{ds} - c\frac{dX}{dy}\frac{dx}{ds} + e\left(\frac{dX}{dy}\frac{dy}{ds} - \frac{dX}{dx}\frac{dx}{ds}\right) = \lambda X.$$

But if we integrate over the whole area of any closed surface, so that x and y are capable of unlimited increase or decrease, this condition is not necessary. We have merely the conditions that X must be one valued and finite over the whole surface.

40. In the same way, if we start with a more general form for the quadric U, we may arrive at equations of a higher order, which it is sufficient that X should satisfy. Thus we may take the additional terms

$$f(X_{22}-X_{12}-X_{21}+X_{11}) + g(X_{21}-X_{11})(X_{13}-2X_{12}+X_{11})$$
$$+ h(X_{12}-X_{11})(X_{31}-2X_{21}+X_{11}) + k(X_{13}-2X_{12}+X_{11})^2$$
$$+ l(X_{31}-2X_{21}+X_{11})^2.$$

These give terms in the differential equation of the fourth order.

Generally, the differential equation which X must satisfy is of the form

$$\Sigma \frac{d^n}{dx^\alpha dy^{n-\alpha}}\left(P\frac{d^nX}{dx^\alpha dy^{n-\alpha}}\right)$$

$$+\Sigma\left[\frac{d^\beta}{dx^\beta}\left(Q\frac{d^\gamma X}{dy^\gamma}\right)+(-1)^{\beta+\gamma}\frac{d^\gamma}{dy^\gamma}\left(Q\frac{d^\beta X}{dx^\beta}\right)\right]=pA^2X,$$

where P, Q, and A are any functions of x, y, and the summations apply to any values of n, α, β, γ we please.

41. Let us now take the whole surface of any sphere as the area of integration. Let the coordinates of any point be given by the usual angular coordinates θ, ϕ; then we may put $x=\cos\theta$, $y=\phi$. Let X be any one valued finite function of xy which satisfies the differential equation of Art. 38. Now x is restricted to lie between ± 1, while $y=\phi$ is unrestricted. Describe an indefinitely small circle whose centre is the origin from which θ is measured. Then along the circumference of this circle, *i.e.*, when $x=-1$, we must have, by Art. 39,

$$b\frac{dX}{dx}+e\frac{dX}{dy}=\lambda X.$$

A similar condition holds at the other limit, *i.e.*, when $x=+1$. Here the constants λ and μ are at our disposal, and in some cases, by giving to them different values, we may obtain different sets of values of p, and therefore different sets of functions X. Let the values of p, for one set thus found, be p, q, ..., and let X, Y, &c. be the corresponding values of X. Then we have

$$\iint A_x^2 XY\,dx\,dy=0;$$

and any one valued finite function of θ, ϕ can be expanded in a series of the form $f(\theta,\phi)=\alpha X+\beta Y+\ldots.$

42. The case of Laplace's functions is included in this proposition, if we can discover the proper values of p, q, &c. *Taking, then, the equation*

$$\frac{d}{dx}\left\{(1-x^2)\frac{dX}{dx}\right\}+\frac{1}{1-x^2}\frac{d^2X}{d\phi^2}+pX=0,$$

we have to determine what values of p will satisfy the conditions at the limits.

First, we notice that the independent variable ϕ has no limits. This is the case considered in Art. 33. We can therefore expand X in a series of the form

$$X=\Sigma\,(u\cos k\phi+v\sin k\phi),$$

with *both* sines and cosines in the expansion. Here u and v are functions of θ. Also, there is but one such expansion for X.

43. Next, we notice that $x = \cos\theta$ is restricted to lie between -1 and $+1$. Exactly as in Art. 34, we must have a condition of the form

$$(1-x^2)\frac{dX}{dx} = \lambda X$$

true at each limit. Here λ is a constant at our disposal

If we substitute the expansion for X in the differential equation, we have

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \left(p - \frac{k^2}{1-x^2}\right)u = 0.$$

We may equate the coefficients of $\cos k\phi$ to zero, because there is but one expansion for zero in a series of sines and cosines of multiples of ϕ which is to be true for *all* values of ϕ. If $P = P$ be a solution of

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + pP = 0,$$

we find, by Leibnitz's theorem, that one value of u is

$$u = (1-x^2)^{\frac{1}{2}k}\frac{d^kP}{dx^k}.$$

If this solution be called $f(x)$, the general solution is

$$u = Lf(x)\int\frac{dx}{f(x)\,(1-x^2)} + Mf(x).$$

Now, if we choose $\lambda = 0$, this value of u must make

$$(1-x^2)\frac{dX}{dx} = 0,$$

when $x = -1$ for all values of ϕ. By the same reasoning as in Art. 35, this is impossible unless $L = 0$. It does not appear that any other choice for λ will permit the condition to be satisfied.

44. Again, when $k = 0$, the equation to find u reduces to the equation considered in Art. 35, and this does not admit of a solution which satisfies the condition at the upper limit except p is of the form $n\,(n+1)$. None of these values of p make $\frac{du}{dx}$ or $\frac{dv}{dx}$ infinite, and are not therefore further limited. We therefore conclude that

$$p = n\,(n+1).$$

45. The general value of X is, under these circumstances, given by the form

$$X = a_0P_n + (a_1\cos\phi + b_1\sin\phi)\,(1-x^2)^{\frac{1}{2}}\frac{dP_n}{dx} + \ldots$$

This value of X has $2n+1$ undetermined constants. This indeterminateness, by Art. 7, shows that the equation to find p has $2n+1$ roots equal to $n\,(n+1)$.

The root $p=0$ occurs once, and makes

$$X = P_n.$$

The root $p=1$ occurs three times; we have therefore

$$Y = a_0 P_1 + (a_1 \cos\phi + b_1 \sin\phi)(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx},$$

$$Z = a_0' P_1 + (a_1' \cos\phi + b_1' \sin\phi)(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx},$$

$$W = a_0'' P_1 + (a_1'' \cos\phi + b_1'' \sin\phi)(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx}.$$

By Art. 30, we have the relations

$$a_0 a_0' + a_1 a_1' + b_1 b_1' = 0,$$
$$a_0 a_0'' + a_1 a_1'' + b_1 b_1'' = 0,$$
$$a_0' a_0'' + a_1' a_1'' + b_1' b_1'' = 0;$$

all these are satisfied by taking

$$Y = P_1,$$

$$Z = \cos\phi\,(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx},$$

$$W = \sin\phi\,(1-x^2)^{\frac{1}{2}} \frac{dP_1}{dx}.$$

The root $p=2$ occurs five times, and may be treated in the same way.

46. We therefore infer that any function of θ, ϕ which is finite, and one valued, may be expanded in a series of the form

$$f(\theta, \phi) = \alpha X + \beta Y + \gamma Z + \ldots,$$

where $\alpha, \beta, \gamma, \ldots$ are constants whose values may be found in the usual way, and are given in Art. 17.

If we collect together all the terms which arise from any one value of p, say $p = n\,(n+1)$, we find

$$P_n \frac{\iint P_n f\,dx\,d\phi}{\iint P_n^2\,dx\,d\phi} + \cos\phi\,(1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} \frac{\iint \cos\phi\,(1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} f\,dx\,d\phi}{\iint \left[\cos\phi\,(1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx}\right]^2 dx\,d\phi} + \&c.$$

Suppose we write $x'\phi'$ for $x\phi$ inside the integrals, then this may be written

$$\iint Q_n f(x'\phi')\,dx'\,d\phi',$$

where

$$Q_n = \frac{P_n P_n'}{\iint P_n^2\,dx\,d\phi} + \frac{\frac{dP_n}{dx}\frac{dP_n'}{dx'} \cos\phi \cos\phi' (1-x^2)^{\frac{1}{2}} (1-x'^2)^{\frac{1}{2}}}{\iint \left[\cos\phi\,(1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx}\right]^2 dx\,d\phi} + \&c.$$

It is not difficult to show that Q_n is the expansion of a Laplace's co-

efficient, which is symmetrical about the axis whose angular coordinates are $\theta\phi$, in a series of the form

$$aP_n + (a_1 \cos\phi + b_1 \sin\phi)(1-x^2)^{\frac{1}{2}} \frac{dP_n}{dx} + \ldots,$$

where P_n is a Laplace's coefficient about the axis from which θ is measured.

Thus, the expansion arrived at for the expansion of $f(\theta, \phi)$ in Laplace's functions agrees with that usually given.

June 12th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. R. C. Rowe, B.A., Fellow of Trinity College, Cambridge, was proposed for election.

The following communications were made:—

"Notes on the Reduction of a System of Forces; and on Plane Curves:" by Mr. J. J. Walker.

"Notes on Determinants of n Dimensions:" by Mr. H. W. Lloyd Tanner.

"Curves for the Inscription of a regular Nonagon and Undecagon in a Circle:" by Rev. T. J. Freeth, LL.D.

"On Clifford's Graphs, and on the Twenty-one Coordinates of a Conic in Space:" by Dr. Spottiswoode, P.R.S.

"Two Geometrical Notes:" by Prof. H. J. S. Smith, F.R.S.

The following presents were received:—

"Educational Times," June, 1879.

"Journal of Institute of Actuaries," Vol. xxi., Pt. v., No. cxv., April, 1879.

"List of Fellows of the Meteorological Society," May 1, 1879.

Notes on Determinants of n Dimensions.

By H. W. Lloyd Tanner, M.A.

[*Read June 12th*, 1879.]

1. *Definition.*—An n-dimensional determinant of the r^{th} order is a function of r^n elements, such as

$$a_{ij\ldots k} \text{ or } (ij \ldots k), \text{ where } i, j \ldots k = 1, 2 \ldots r;$$

each element being designated by n symbols $i, j \ldots k$.* It is the sum of

* We shall have to refer so frequently to these symbols, that it is desirable to give them a special name. In the paper they are called "sub-elements"; a name which may be taken to allude to their position in the signature of the element $a_{ij\ldots k}$, or to indicate that each element is, in a certain sense, made up of its sub-elements.

all the different terms—each with its proper sign—that can be formed from the elements by the following process:—Take r elements in which i is replaced by $1, 2 \ldots r$ respectively, in this order; j is replaced by $1, 2 \ldots r$ in an arbitrary order; the whole range $1, 2 \ldots r$ is to be employed so that no repetitions occur: and so on, until the signature of all the elements is complete. The product of these elements is a term of the determinant; and of these terms there are $(r!)^{n-1}$. The sign of the term is thus defined. The sub-elements of the second rank can be brought to the order $1, 2 \ldots r$ by a certain number of transpositions; and so can the sub-elements of each rank in turn. When the whole number of transpositions thus required is even, the term is positive; when the number is odd, the term is negative. A rule will, however, be given in the sequel, by which the sign of any term can be found without going through the long process just described.

2. *Notation.*—The topographical notation commonly employed for ordinary determinants, though it suggested the title of the paper, cannot be used for determinants of higher dimensions. It may conveniently be replaced by an obvious extension of Sylvester's umbral notation. The symbol of a determinant is

$$a \left| \begin{matrix} 1, 2 \ldots r \\ 1, 2 \ldots r \\ \ldots\ldots\ldots \\ 1, 2 \ldots r \end{matrix} \right| \ (n \text{ rows}).$$

The top row consists of sub-elements of the first rank; the second row gives those of the second rank; and so on. The a on the left is only needed when we wish to distinguish between different determinants. In most cases it is sufficient to write the symbol in the more compact form

$$a \mid 1, 2 \ldots r \mid^n,$$

where the exponent n indicates the dimensions of the determinant.

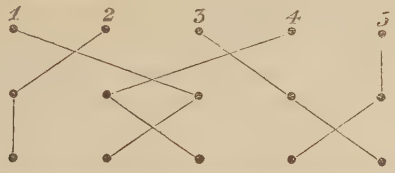
Any element of a determinant may be denoted by drawing a line in the symbol of the determinant through the sub-elements in succession. In practice it is convenient to omit the numbers representing the sub-elements, or only to write the top row.

By a group of these lines—say "elemental lines"—a term may be represented. For instance, the term

$$132 \,.\, 211 \,.\, 345 \,.\, 423 \,.\, 554$$

In the element just written, i is said to be of the first rank, j of the second rank, and so on. Two sub-elements, belonging to different ranks, are to be regarded as different sub-elements, even though they happen to be represented by the same symbol. Thus, in the determinant of the r^{th} order of n dimensions, there are $r \,.\, n$ different sub-elements.

is indicated by the diagram

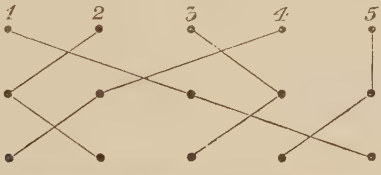


Evidently, when the term is complete, every sub-element in the diagram has one, and only one, elemental line passing through it.

3. A term may be represented by another diagram, or, rather, group of diagrams. Instead of connecting the sub-elements of the first rank with those of the second rank, these with the third rank, and so on, in succession; we connect the first rank with each of the other ranks separately. We thus obtain $n-1$ separate diagrams, one for each rank except the first, and the whole may be regarded as a diagram of the term. [In the case below figured we have joined these separate diagrams together.] The diagram of the term

$$132.211.345.423.554$$

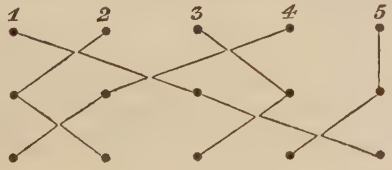
is



The upper part is the same as that of the diagram of Art. 2; the lower part consists of lines joining the sub-elements of the first rank to those of the third rank, belonging to the same elements; viz., 1 to 2, 2 to 1, 3 to 5, 4 to 3, and 5 to 4.

The diagram just described has the following property:—The number of intersections in it is equal to the number of transpositions required to reduce the sub-elements of the several ranks to the order $1, 2 \ldots r$; therefore, by the definition of Art. 1, the term is positive or negative according as this number is even or odd. Moreover, the diagram shows what transpositions are required to complete the reduction.

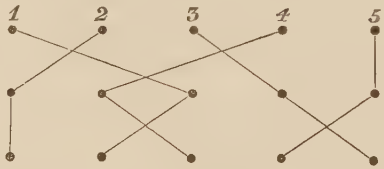
To prove this, suppose that every intersection is cut vertically, and that then the two lines on the left of the cut are joined together, and likewise those on the right. The diagram of this article would thus become



Every cut made is evidently equivalent to a transposition of two sub-elements. For, suppose two intersecting lines run down to the sub-elements i, j; then the line that originally went to i, will, after the intersection is removed, go to j, and *vice versâ*. Hence the change in the diagram corresponds to a transposition of the two sub-elements i, j. Also, it is clear from the diagram that, when all the intersections are removed, the sub-elements of each rank are reduced to the order $1, 2 \ldots r$. Thus, if there be p intersections in a diagram of a term, the sub-elements of the several ranks can be reduced to the natural order by p transpositions; and the term is therefore positive or negative according as p is even or odd. For instance, the term figured in this article is positive, for there are six intersections in its diagram.

The safest way of counting the intersections in a complicated diagram, is to count them along each of the lines in the diagram: the number thus obtained is $2p$; for each intersection is reckoned twice. The risk of omitting an intersection, or of counting one twice, is thus avoided; and the circumstance that the number obtained should be even ($2p$) serves as an additional check.

4. The diagram of Art. 2 may also be made to give the sign of the term to which it belongs, though not by simply counting the intersections. Indeed, a reference to the figure shows that, whilst a transposition of two sub-elements in one of the extreme ranks (the first or last) changes the number of intersections in the diagram by an odd number, a similar transposition in an intermediate rank changes the number of intersections by an even number. For instance, in the diagram



if we transpose 1, 2 in the top row, or 2, 3 in the bottom, one intersection is lost; if we transpose 2, 3 in the top row, or 3, 4 in the bottom row, one intersection is gained. But with the second rank it is different. If 2, 3 be transposed, *two* intersections are lost; if 3, 4, *two* intersections are gained; while, by interchanging 4, 5, no change is made in the number of intersections. This arises from the fact that a transposition of two sub-elements of an intermediate rank affects the portions of the elemental lines immediately above, and those immediately below; it affects the diagram doubly. It is suggested by this, that if we omit the portion of the diagram above or below an intermediate rank (so that every sub-element is approached, on one side only, by an elemental line), the remaining intersections will correctly determine the sign of

the term. And, in fact, it is so. If we retain only those portions of the diagram lying between the n^{th} and $\overline{n-1}^{\text{th}}$ ranks, the $\overline{n-2}^{\text{th}}$ and $\overline{n-3}^{\text{th}}$ ranks, and so on, and omit every alternate section, the reduced diagram will include an even or odd number of intersections according as the term is positive or negative. For instance, in the diagram last drawn, we omit the upper half. The remainder contains two intersections, and the term is therefore positive.

To prove the rule, consider a portion of the reduced diagram lying between two consecutive ranks. Suppose that the order of the sub-elements in the upper rank is reducible to the natural order in q transpositions. Also, let the number of intersections in the portion of the diagram considered be k. Then, by the reasoning of Art. 3, we infer that the order of the lower rank can be reduced to that of the upper rank by k transpositions, and therefore to the natural order by $q+k$ transpositions. The sum of these is $2q+k$. Deal with all parts of the reduced diagram, and sum the results. We thus get a number which, by Art. 1, is even or odd according as the term is positive or negative. But the result is not affected if we only take into account k; and this is what we do in the process of the present article.

Or the matter may be put thus:—Let $p_1, p_2 \ldots p_{n-1}$ be the number of intersections between the first and second, the second and third, and the $\overline{n-1}^{\text{th}}$ and n^{th} ranks of the complete diagram. Then the order of the sub-elements of the second rank is reducible to that of the first by p_1 transpositions; the third to the second by p_2; and therefore to the first by p_1+p_2 transpositions, and so on. Hence the whole set is reducible to the natural order by

$$p_1+(p_1+p_2)+(p_1+p_2+p_3)+\ldots+(p_1+p_2+\ldots+p_{n-1})$$

or

$$p_{n-1}+2p_{n-2}+3p_{n-3}+\ldots+(n-2)\,p_2+(n-1)\,p_1$$

transpositions; and the term is positive or negative according as this number is even or odd. But in this expression we may omit the 2nd, 4th, &c. terms, for, being even, they do not affect the result; also, we may replace the coefficients of the 3rd, 5th, &c. terms by unity, since this is equivalent to subtracting an even number from the expression. When these reductions are made, there remains just the rule which was to be verified.

5. From the definitions and explanations already given, it is easy to deduce many properties in which determinants of n dimensions resemble ordinary determinants.

Every sub-element enters once, and only once, into a term. Hence a determinant is a linear and homogeneous function of the elements which have a sub-element in common. Just as an ordinary determinant

is a linear homogeneous function of the elements in any row, or in any column.

If the elements with a sub-element in common are expressible as the sum of p terms, the determinant is also expressible as a sum of p determinants, in each of which each of the elements in question is replaced by one of its terms. If all the elements having a sub-element in common be multiplied by the same factor, the whole determinant is so multiplied. In particular, if all these elements vanish, so does the determinant. [It is convenient in this case to replace the sub-element by 0, and then we may say that a determinant with a vanishing sub-element in its symbol, also vanishes.] These results flow from the first proposition of the article, just as in the case of ordinary determinants.

From Art. 1, it appears that, by transposing two sub-elements of the same rank, the sign of the term is altered. Hence the sign of the whole determinant is reversed by an interchange of two subscripts of the same rank. From this, again, it appears that a determinant with two identical sub-elements in the same rank vanishes. By a somewhat bold but very convenient figure of speech, we may say that the value of a determinant is not altered when to any sub-element is added a linear function of the sub-elements in the same rank. What is meant may be seen by an example. We will suppose the sub-element is 1 in the last rank ; and take α, β, &c. groups of sub-elements of the first $n-1$ ranks, such that $(\alpha 1)$, $(\beta 1)$, &c. represent all the elements that have in common the sub-element 1 of the last rank. The theorem is, that if we replace $(\alpha 1)$, $(\beta 1)$, &c. by

$$(\alpha 1)+\lambda_2(\alpha 2)+\lambda_3(\alpha 3)+\ldots+\lambda_r(\alpha r),$$

$$(\beta 1)+\lambda_2(\beta 2)+\lambda_3(\beta 3)+\ldots+\lambda_r(\beta r),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

respectively, the value of the determinant is not changed.

These propositions are analogous to the theorems—that, by transposing two rows, or two columns, of an ordinary determinant, the sign is changed ; that, when two rows, or two columns, are identical, the determinant vanishes; that, if to any line (row or column) we add a linear function of the parallel lines (still the same figure of speech), the value of the determinant is not changed.

6. Another property requires a closer examination. An ordinary determinant is not affected by interchanging its rows and columns ; or, in other words, by a transposition of the two lines of its umbral symbol. The effect of this transformation upon any particular term is to change it to another with the same sign; and thus the effect

upon the whole determinant is merely to change the order in which the terms occur. For, suppose the determinant to be

$$A_1+A_2+\ldots+A_\mu-B_1-B_2-\ldots-B_\mu.$$

If every term A is transformed into an A, and therefore every B into a B, the determinant is not altered. If every A is changed into a B, and therefore every B into an A, the determinant becomes

$$B_i+B_j+\ldots-A_k-A_l-\ldots,$$

and it is changed in sign; but otherwise its value is not altered. If some A's are changed into A's, and some into B's, the determinant takes the form $A_i+B_j+\ldots-A_k-B_l-\ldots,$

and its value is altogether changed.

If we leave the first rank out of the question, we can prove, from the definition in Art. 1, that a transposition of two ranks of sub-elements does not change the value of the determinant. For, in every term, each rank has a certain number belonging to it; and the sign of the term depends upon the sum of these numbers. Suppose, now, two ranks to be interchanged. In general, the term is changed; but the sign is still the same, for the sum of the rank-numbers is not changed by adding them up in a different order. Hence we have the first of the three cases noted above; every A is changed into an A, every B into a B, and the value of the determinant is unchanged.

It remains to notice the effect of interchanging the first rank of sub-elements with one of the others. For convenience, we suppose the rank that is to be brought to the top, first transposed with the second rank. This, as we have just seen, does not change the value of the determinant, and it enables us to confine our attention to the effect of transposing the first and second ranks. We select any term, and draw its diagram as in Art. 2. Using the notation of Art. 4 (last paragraph), we know that this term is positive or negative as the expression

$$(n-1)\,p_1+(n-2)\,p_2+\ldots+p_{n-1}\ (=P_1, \text{ say})$$

is even or odd.

Now, examine the sign of the term formed by transposing the first and second ranks. The number of transpositions required to reduce the order of the sub-elements of the first rank to that of the second (we use the original sequence of the ranks throughout) is p_1. To reduce the third rank to the second requires p_2 transpositions; to reduce the fourth requires p_2+p_3; and so on. Hence, to reduce the several ranks to the same order as the second, requires

$$p_1+p_2+(p_2+p_3)+\ldots+(p_2+p_3+\ldots+p_{n-1}),$$

or $$p_1+(n-2)\,p_2+(n-3)\,p_3+\ldots+p_{n-1}\ (=P_2, \text{ say})$$

transpositions; and the term is positive or negative according as P_2 is even or odd. Hence the sign of the original term, and that of the one into which it is transformed, are the same or different according as

$$P_1 - P_2, = (n-2)\,p_1,$$

is even or odd.

When n is even, this expression is always even; so that, in this case, the original and the transformed terms have the same sign; and the determinant is not changed by the transposition. Hence, *a determinant of even dimensions is not changed when any two ranks of its sub-elements are transposed.* This includes the case of ordinary determinants.

When n is odd, the expression $P_1 - P_2$ is even or odd as p_1 is even or odd. But p_1 is even for half the terms of the determinant, and odd for the other half. That is to say, in determinants of odd dimensions, when the first rank of sub-elements is changed, half of the terms are transformed into terms of the same sign, and half are transformed into terms of the opposite sign. Of course, by such a change, the value of the determinant is altered. Hence, *a determinant of odd dimensions changes its value when its first rank of sub-elements is changed; but not when any two other ranks are transposed.*

The result is so unexpected that it may be well to verify it by an example. Take the determinant

$$|\,1,\,2\,|^3.$$

Upon development, this is

$$111\,.\,222 - 112\,.\,221 - 121\,.\,212 + 122\,.\,211.$$

On transposing the second and third sub-elements, this becomes

$$111\,.\,222 - 121\,.\,212 - 112\,.\,221 + 122\,.\,211,$$

the same as before, in accordance with the rule. On transposing the first and second elements, we get

$$111\,.\,222 - 112\,.\,221 - 211\,.\,122 + 212\,.\,121;$$

and, from an interchange of the first and third sub-elements, we get

$$111\,.\,222 - 211\,.\,122 - 121\,.\,212 + 221\,.\,112.$$

These last results differ from each other, and from the original determinant; so that, in this particular instance, our conclusion is verified.

It is an immediate consequence of the above, that a determinant of n dimensions, n being odd, has n different values. It has no more, for when the first rank of sub-elements is chosen, the value is determinate: all the n values are different, for otherwise we could, in some cases, change the first rank of sub-elements without altering the value of the determinant.

7. We may also regard these facts from another point of view. Consider a term of a determinant. Suppose that n_1 ranks are of one class as regards the sequence of the sub-elements; and n_2 ranks are of the other class.* If any one of the n_1 ranks be taken as the first rank, the sign of the term is $(-)^{n_2}$; if any of the n_2 ranks be taken as first rank, the sign of the term is $(-)^{n_1}$. These signs are the same when n_1+n_2 (or n) is even; for then n_1, n_2 are both even, or both odd. On the other hand, the signs are different when n is odd, for then one of the numbers n_1, n_2 must be odd, and the other even. Thus, generally, a term of a determinant of odd dimensions may be either positive or negative; the only exception being when n_1 is 0.

Let a determinant of n dimensions, in which the first rank has been fixed, be developed. From it select those terms in which the i^{th}, j^{th}, &c. ranks (and only these) are of the opposite class to the first rank. Indicate the aggregate of ese selected terms by the symbol

$$S_{ij\ldots}.$$

Such an aggregate may be termed a section of the determinant; and all the terms in it are positive or negative according as the number of subscripts is even or odd. We propose to examine the effect of a change of the first rank upon one of these sections.

When the first rank is transposed with a rank of the same class as itself, *i.e.*, with a rank that does not appear amongst the subscripts to $S_{ij\ldots}$, the section is not changed, for the i^{th} rank is still of the opposite class to the leading rank, and so on.

When the first rank is interchanged with (say) the i^{th} rank, the section is changed. Every one of range 2, 3 ... n, not included in the subscripts to $S_{ij\ldots}$, appears among the subscripts of the transformed section (S'); for, if the k^{th} rank is of the same class as the (original) first rank, it must be of the opposite class to the (original) i^{th} rank; that is, if k is not among the subscripts to S, it is amongst those to S'. Also, the subscript i will appear in the signature of S'. As to the number of subscripts, let those of S be n_1, those of S', n_2.

Then
$$n_1+n_2=n;$$

for we exhaust the range 2, 3 ... n, and i occurs twice. If, then, n be even, the section is not changed in sign; if n be odd, the section is changed to one which had the opposite sign.

For instance, let Δ_1, Δ_2, Δ_3 represent three values of a determinant of three dimensions; Δ_2, Δ_3 being obtained from the Δ_1 by transposing

* Two permutations are, or are not, of the same class, according as they can, or cannot, be reduced to the same order by an even number of transpositions.

the first with second or third ranks respectively. Then

$$\Delta_1 = S - S_2 - S_3 + S_{23}.$$

On transposing the first and second ranks, S_2 becomes S_{23} (a term of opposite sign), and S_{23} becomes S_2; thus

$$\Delta_2 = S - S_{23} - S_3 + S_2.$$

But it is evident that we may attain the value of Δ_2 without seeking the section into which any particular S is changed; it is only necessary to change the sign of every S with a 2 in its signature, leaving the order unchanged. Thus

$$\Delta_1 = S - S_2 - S_3 + S_{23},$$

$$\Delta_2 = S + S_2 - S_3 - S_{23};$$

similarly

$$\Delta_3 = S - S_2 + S_3 - S_{23}.$$

It is easy to show that a determinant of n dimensions has 2^{n-1} sections, each of which is composed of $(3, 4 \ldots r)^{n-1}$ terms.

8. Any determinant can be expressed as a sum of determinants of lower dimensions. For instance,

$$|123|^3 = \left|\begin{matrix}11, 22, 33\\ 1, 2, 3\end{matrix}\right| - \left|\begin{matrix}11, 23, 32\\ 1, 2, 3\end{matrix}\right| + \left|\begin{matrix}12, 23, 31\\ 1, 2, 3\end{matrix}\right|$$

$$- \left|\begin{matrix}12, 21, 33\\ 1, 2, 3\end{matrix}\right| + \left|\begin{matrix}13, 21, 32\\ 1, 2, 3\end{matrix}\right| - \left|\begin{matrix}13, 22, 31\\ 1, 2, 3\end{matrix}\right|$$

The sign of each determinant on the right is that of its principal term considered as a term of the determinant on the left; or, what is the same thing, it is the sign of the top (double) row considered as a development of the first two ranks of the determinant on the left.

Generally, $|1, 2 \ldots r|^n$ can be expressed as a sum of $r!$ determinants of $n-1$ dimensions. For, we may form a double rank by joining the r sub-elements of the first rank to those of the second, one to one. The same process may be repeated upon each of the determinants thus obtained; and then again as often as we please. Thus a determinant of the r^{th} order and of n dimensions can be expressed as a sum of $(r!)^{n-m}$ determinants of the r^{th} order and of m dimensions. In particular, it can be reduced to an aggregate of $(r!)^{n-2}$ ordinary determinants; and it consists of $(r!)^{n-1}$ terms, each of which is the product of r factors.

It is important to notice that the development of a determinant of the r^{th} order in terms of determinants of one lower dimension, is the same whatever the dimensions may be; for the distinguishing double rank, and the sign of each of the lower determinants, are derived from the first two ranks only of the original determinant, and are not

affected by the number of ranks that follow. Of course, the same is true of the reverse process of expressing a group of determinants as one of higher dimensions.

9. From any determinant of the r^{th} order may be formed a minor determinant of the m^{th} order, by selecting any m sub-elements of the first, second, and other ranks in turn. The remaining sub-elements form a minor of the $\overline{r-m}^{\text{th}}$ order, which may be called conjugate to the first when its sign is properly determined. This is to be the sign of the product of the principal terms of the two conjugate minors, considered as a term of the original determinant. For instance,

$$\begin{vmatrix} 3, 5 \\ 4, 5 \\ 5, 4 \end{vmatrix} \text{ and } \begin{vmatrix} 2, 1, 4 \\ 1, 3, 2 \\ 2, 3, 1 \end{vmatrix}$$

are conjugate minors of $|1, 2, 3, 4, 5|^3$ because

$$345 \,.\, 554 \,.\, 212 \,.\, 133 \,.\, 421$$

is a positive term of this determinant.

If the order and dimensions of the original determinant be r, n respectively, there are

$$\left\{ \frac{r\,!}{m\,!\; r-m\,!} \right\}^n$$

minors of the m^{th} order; and, of course, the same number of minors of the $\overline{r-m}^{\text{th}}$ order. For, we may choose any m of the r sub-elements in each rank, different arrangements not being counted. If we use m specified sub-elements of the first rank (for instance, 1, 2 ... m), there will be only

$$\left\{ \frac{r\,!}{m\,!\; r-m\,!} \right\}^{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

minors.

Form all minors of the m^{th} order on the hypothesis that the first rank of each consists of the same m sub-elements, and multiply each by its conjugate. The sum of all the products thus obtained is the original determinant. For, the product of two conjugate minors consists of

$$\{m\,!\; r-m\,!\}^{n-1}$$

terms, each of which is a term of the original determinant. Taking the sum of all the products, we shall have

$$\{r\,!\}^{n-1}$$

terms, the number of products being that marked (1) above; and this is just the number of terms in the original determinant. It remains to show that, amongst these products, no terms are repeated, and this we

can do by proving that every term of the determinant is included in the sum. To do this, take any term of the determinant, and draw its diagram as in Art. 2. The elemental lines which start from the m selected sub-elements of the first rank, pass through mn sub-elements, of which we can form a minor of the m^{th} order. The product of this with its conjugate evidently includes the term in question. Hence we have two expressions, the developments of which consist of an equal number of terms, and such that every term of the one is included in the other. Clearly, these must be identically equal.

Of course, the process may be repeated, so that any determinant may be represented as a sum of products of 2, 3 ... r minor determinants, each of n dimensions; r, n being the order and dimensions of the original determinant.

10. In the symbols hitherto used, the number of sub-elements in a rank has been the same throughout. When this is not the case, as, for instance, in the symbol

$$\begin{vmatrix} 1, 2, \ldots p \\ 1, 2, \ldots\ldots q \\ \ldots \quad \ldots \quad \ldots \\ 1, 2, \ldots\ldots\ldots r \end{vmatrix},$$

we have, not a determinant, but a system of determinants. These determinants are the minors of the p^{th} order of the determinant

$$\begin{vmatrix} 1, 2, \ldots p, 0 \ldots 0 \\ 1, 2, \ldots\ldots q \ldots 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ 1, 2, \ldots\ldots\ldots\ldots r \end{vmatrix},$$

where the places of the deficient sub-elements are filled up by 0's; and it is assumed that every determinant, with a 0 amongst its sub-elements, vanishes. The number of determinants in the system is

$$\frac{q!}{p!\,q-p!} \ldots\ldots \frac{r!}{p!\,r-p!}.$$

It is assumed, as in ordinary matrices, that

$$p \geqslant q \geqslant \ldots\ldots \geqslant r.$$

11. The result of multiplying together two determinants may be expressed in a theorem analogous to that which obtains in the case of ordinary determinants. It is convenient to assume that the two determinants are of the same order. Should this not be the case, it is easily made so by adding elements to the determinant of the lower

order. Every supplementary element of the form $(ii \ldots i)$ is unity; all the others zero.

The theorem is this:—

$$\left.\begin{aligned} a\,|\,1, 2, \ldots r\,|^m \times b\,|\,1, 2, \ldots r\,|^n &= c\,|\,1, 2, \ldots r\,|^{m+n-2} \\ \text{where} \qquad c_{i\ldots jk\ldots l} &= \sum_{p=1}^{p=r} a_{i\ldots jp}\, b_{k\ldots lp} \end{aligned}\right\} \ldots(2).$$

[The ranges $i \ldots j$ and $k \ldots l$ respectively comprise $m-1$ and $n-1$ sub-elements.]

This may be proved inductively. Assuming it to be true for $m-1, n$, it can be shown to hold good for m, n. Hence, knowing it to be true for two ordinary determinants, we know it to be generally true.

Let the a-determinant be expressed as a sum of determinants of $m-1$ dimensions, as in Art. 8; and suppose one of these determinants to be

$$\epsilon \,.\, a \begin{vmatrix} 1i, & 2j, & \ldots & rk \\ 1, & 2, & \ldots & r \\ \ldots & \ldots & \ldots & \end{vmatrix} (m-1 \text{ rows}),$$

ϵ being $+1$ or -1.

The product of this by the b-determinant is

$$\epsilon \,.\, c \begin{vmatrix} 1i, & 2j, & \ldots & rk \\ 1, & 2, & \ldots & r \\ \ldots & \ldots & \ldots & \end{vmatrix} (m+n-3 \text{ rows}),$$

where c is defined as in (2), if we assume the theorem (2) to be true for $m-1, n$. But the a-determinants (3) taken together form a single determinant of one higher dimension; and, as the c-determinants (4) differ from the a-determinants (3) only by having more ranks, the same is true of them (Art. 8). Let the a-determinants and the c-determinants be combined into a single a-determinant and a single c-determinant of dimensions higher by one. Then (2) is reproduced. Or, the theorem is proved for m, n when it is assumed for $m-1, n$. Now, it is true for two ordinary determinants (2, 2); and therefore, from successive applications of the result just obtained, it is proved generally true.

12. In the preceding article the variable sub-element p has been assumed to be of the last rank in each of the determinants a, b. But, since the ranks may be transposed, the theorem (2) will still be true if p be in any position in the signature of the a-elements, except the first position, when m is odd. Of course, p must be of the same rank in all the elements. Similarly, the rank of p in the b-elements may be any

except the first, when n is odd. Hence we may enunciate the multiplication theorem thus,

$$a|1, 2, \ldots r| \times b|1, 2, \ldots r|^n = c|1, 2, \ldots r|^{m+n-2},$$

where $$c_{i\ldots jk\ldots l} = \sum_{p=1}^{p=r} a_{i\ldots p\ldots j}\, b_{k\ldots p\ldots l};$$

the p being the s^{th} subscript to a, and the t^{th} subscript to b. But s must not be 1 when m is odd; nor may t be 1 when n is odd.

Notes on Plane Curves. By J. J. Walker, M.A.

[*Read June 12th*, 1879.]

I. *Reduction of a Ternary Cubic to the sum of Four Cubes.*

Some time back, when occupied about another question, I was led to observe the method of actually reducing a ternary cubic to the quadrinomial form $ax^3+by^3+cz^3+dw^3=0$ .....(1), which has not, as far as I am aware, been given by those geometers who have employed this form. It may be worth while to give it.

Let $x'y'z'$ be any point on the right line

$$lx+my+nz=0 \text{ .....(2)};$$

its polar conic with respect to a cubic, $u=0$, is

$$x'\frac{du}{dx}+y'\frac{du}{dy}+z'\frac{du}{dz}=0,$$

which, since $x'y'z'$ satisfy (2), may be written

$$y'\left(l\frac{du}{dy}-m\frac{du}{dx}\right)+z'\left(l\frac{du}{dz}-n\frac{du}{dx}\right)=0,$$

showing that it passes through the four points common to

$$n\frac{du}{dy}-m\frac{du}{dz}=0, \quad l\frac{du}{dz}-n\frac{du}{dx}=0, \quad m\frac{du}{dx}-l\frac{du}{dy}=0;$$

i.e., if $$u \equiv a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3a_3x^2z+3b_1y^2x+3b_3y^2z + 3c_1z^2x+3c_2z^2y+6dxyz,$$

through the four points common to

$$(na_2-ma_3)\,x^2+(nb_2-mb_3)\,y^2+(nc_2-mc_3)\,z^2+2\,(nb_3-mc_2)\,yz$$
$$+2\,(nd-mc_1)\,zx+2\,(nb_1-md)\,xy=0,$$

$$(la_3-na_1)\,x^2+(lb_3-nb_1)\,y^2+(lc_3-nc_1)\,z^2+2\,(lc_2-nd)\,yz$$
$$+2\,(lc_1-na_3)\,zx+2\,(ld-na_2)\,xy=0,$$

$$(ma_1-la_2)\,x^2+(mb_1-lb_2)\,y^2+(mc_1-lc_2)\,z^2+2\,(md-lb_3)\,yz$$
$$+2\,(ma_3-ld)\,zx+2\,(ma_2-lb_1)\,xy=0.$$

If the triangle of reference is that which is self-conjugate with respect to these conics, then the coefficients of yz, zx, xy in the above three equations must vanish simultaneously, for which conditions it is necessary and sufficient that

$$a_2=\frac{ld}{n},\quad a_3=\frac{ld}{n},\quad b_1=\frac{md}{n},\quad b_3=\frac{md}{l},\quad c_1=\frac{nd}{m},\quad c_2=\frac{nd}{l};$$

whence u is of the form

$$u=a_1x^3+b_2y^3+c_3z^3$$
$$+d\left(3\frac{l}{n}x^2y+3\frac{l}{m}x^2z+3\frac{m}{n}y^2x+3\frac{m}{l}y^2z+3\frac{n}{m}z^2x+3\frac{n}{l}z^2y+6xyz\right)=0;$$

or, adding and subtracting

$$d\left(\frac{l^2}{mn}x^3+\frac{m^2}{nl}y^3+\frac{n^2}{lm}z^3\right),$$

$$u=\left(a_1-\frac{l^2d}{mn}\right)x^3+\left(b_2-\frac{m^2d}{nl}\right)y^3+\left(c_3-\frac{n^2d}{lm}\right)z^3$$
$$+\frac{d}{lmn}(l^3x^3+m^3y^3+n^3z^3+3l^2mx^2y+3l^2nx^2z+3m^2ly^2x+3m^2ny^2z$$
$$+3n^2lz^2x+3n^2mz^2y+6lmnxyz);$$

i.e., multiplying by lmn, u is of the form

$$ax^3+by^3+cz^3+dw^3,$$

where $a=(a_1mn-dl^2)\,l$, $b=(b_2nl-dm^2)\,m$, $c=(c_3lm-dn^2)\,n$,

$$w=lx+my+nz.$$

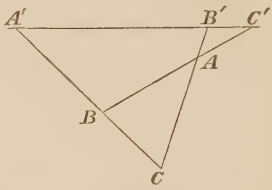
Hence, to reduce a cubic to the sum of four cubes: take any arbitrary line, which determines four points, viz., those common to the polar conics of all points on it; taking this as one of the four lines of reference, the others will be the connectors of the points of intersection of the three pairs of lines passing through the four points so determined.

The Hessian of (1) is

$$d\,(bclyz+camzx+abnxy)(lx+my+nz)-abcxyz;$$

the form of which shows that the corners of the triangle (xyz) lie on it, and that the third points in which the Hessian is met by the sides lie on the line $lx+my+nz$. Thus arises another representation of the reduction: An arbitrary line determines a triangle, inscribed in the Hessian, the three sides of which again meet that curve at the points in which it is cut by the line; the three sides of the triangle, and the line determining it, form the four lines of reference.

But the relation among the three sides of the triangle and the arbitrary line is obviously symmetrical; viz., taking one of those sides as the arbitrary line, then the other three form the triangle, which is self-conjugate with respect to the polar conics of all points on the first. Thus, in the accompanying figure, the line $A'B'C'$, cutting the Hessian in those points, and the sides of the triangle ABC, may be taken as representing the four lines of reference, or the line CAB' with the sides of the triangle $C'A'B$, and so on.



In fact, the points AA', BB', CC' are three pairs of Prof. Cayley's "conjugate poles"; *i.e.*, the line-pair forming the polar conic of one of each pair of points intersect in the other.

Consider the point A $(y=0,\ z=0)$; its polar conic is

$$ax^2+ldw^2;$$

i.e., a pair of lines meeting in the point common to x and w; viz., A'.

Thus, finally, arises another method of determining a system of four lines of reference. Take any line cutting the Hessian in three points; the conjugate poles of these with respect to the cubic form with them a system of six points lying three and three on four lines; which being taken as lines of reference, the equation of the cubic is reduced to the form

$$ax^3+by^3+cz^3+dw^3=0.$$

II. *On a certain Derived Curve.*

In connexion with Cotes's Theorem, the following seems worthy of being recorded:—

Consider a transversal $L\,(lx+my+nz=0)$, cutting a given line $R\,(\alpha x+\beta y+\gamma z=0)$ in the point P, and the curve (of order p) $u=0$ in the p points O_1, O_2 ... O_p; there are $p-1$ points O, O', O''... on L, for which

$$\frac{1}{OO_1}+\frac{1}{OO_2}+\ldots+\frac{1}{OO_p}=\frac{p}{OP},$$

$$\frac{1}{O'O_1}+\frac{1}{O'O_2}+\ldots+\frac{1}{O'O_p}=\frac{p}{OP},$$

... ...

The locus of these $p-1$ points, as the transversal L revolves about a fixed point $x_1y_1z_1$, may be shown to be another curve of order p, passing through $x_1y_1z_1$, and through the $(p-1)^2$ poles of the line R with respect to u; and consequently, when $x_1y_1z_1$ coincides with one of those poles, having it as a double point.

I recall the equation by which the segments on the transversal between (xyz), and the p points in which it meets $u=0$, may be determined ("Proceedings," Vol. IX., p. 227); viz., these will be equal to the values of $\rho\,(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}$, from

$$u+p\rho Du+\frac{p\,.\,p-1}{1\,.\,2}\rho^2D^2u\,\ldots+\rho^pD^pu=0,$$

where $pDu=(m\sin C-n\sin B)\dfrac{du}{dx}+(n\sin A-l\sin C)\dfrac{du}{dy}$

$$+(l\sin B-m\sin A)\frac{du}{dz};$$

from which, first making $u=\alpha x+\beta y+\gamma z=R$, to the same factor *près*,

$$\frac{1}{OP}=\frac{(\beta\sin C-\gamma\sin B)\,l+(\gamma\sin A-\alpha\sin C)\,m+(\alpha\sin B-\beta\sin A)\,n}{R},$$

and $\Sigma\dfrac{1}{OO_1}=$

$$\frac{\left(\dfrac{du}{dy}\sin C-\dfrac{du}{dz}\sin B\right)l+\left(\dfrac{du}{dz}\sin A-\dfrac{du}{dx}\sin C\right)m+\left(\dfrac{du}{dx}\sin B-\dfrac{du}{dy}\sin A\right)n}{u}$$

so that the condition determining O, O', $O''\ldots$, gives

$$\frac{(\beta\sin C-\gamma\sin B)\,l+\ldots}{R}=\frac{(u''\sin C-u'''\sin B)\,l+\ldots}{u},$$

where $u'=\dfrac{1}{p}\dfrac{du}{dx}$, $u''=\dfrac{1}{p}\dfrac{du}{dy}$, $u'''=\dfrac{1}{p}\dfrac{du}{dz}$; or

$$\sin A\,\{(m\gamma-n\beta)\,u-(mu'''-nu'')\,R\}+\sin B\,\{\ldots\}+\sin C\,\{\ldots\}=0.$$

But, since xyz, $x_1y_1z_1$ are points on $lx+my+nz=0$,

$$l:m:n=yz_1-zy_1:zx_1-xz_1:xy_1-yx_1;$$

by which substitutions, and those of $xu'+yu''+zu'''$ for u, $\alpha x+\beta y+\gamma z$ for R, the equation above reduces identically to

$$x\sin A\,\{(\alpha x+\beta y+\gamma z)\,(x_1u'+y_1u''+z_1u''')$$
$$-(\alpha x_1+\beta y_1+\gamma z_1)\,(xu'+yu''+zu''')\}$$
$$+y\sin B\,\{\ldots\}+z\sin C\,\{\ldots\}=0.$$

This, divided by $x \sin A + y \sin B + z \sin C$, gives, as the required locus,

$$v \equiv (\alpha x + \beta y + \gamma z)(x_1 u' + y_1 u'' + z_1 u''') - (\alpha x_1 + \beta y_1 + \gamma z_1)\, u = 0;$$

or, in another form,

$$v \equiv (yz_1 - zy_1)(\gamma u'' - \beta u''') + (zx_1 - xz_1)(\alpha u''' - \gamma u') + (xy_1 - yx_1)(\beta u' - \alpha u'') = 0;$$

which, plainly, passes through the point $(x_1 y_1 z_1)$, and through the $(p-1)^2$ points of intersection of the system

$$\gamma \frac{du}{dy} - \beta \frac{du}{dz} = 0, \quad \alpha \frac{du}{dz} - \gamma \frac{du}{dx} = 0, \quad \beta \frac{du}{dx} - \alpha \frac{du}{dy} = 0.$$

Further, if $x_1 y_1 z_1$ coincides with one of the points common to this system, then it is evidently a double point on v; in fact, for $x = x_1$, $y = y_1$, $z = z_1$,

$$\frac{dv}{dx} = y_1 (\beta u' - \alpha u'') - z_1 (\alpha u''' - \gamma u');$$

and therefore vanishes if $x_1 y_1 z_1$ is one of the points common to

$$\beta \frac{du}{dx} - \alpha \frac{du}{dy}, \quad \alpha \frac{du}{dz} - \gamma \frac{du}{dx};$$

and, similarly, $\frac{dv}{dy}$, $\frac{dv}{dz}$.

III. *Envelope of Diameters of a Cubic.*

In the case of u being a cubic (p. 183),

$$u + 3\rho Du + 3\rho^2 D^2 u + \rho^3 D^3 u = 0$$

determines the intercepts on any transversal between a point xyz in it, and the points in which the transversal meets u, when it is drawn parallel to a fixed line $lx + my + nz = 0$. Hence the equation of the rectilinear *diameter* to a system of chords parallel to $lx + my + nz = 0$, in the sense of the locus of a point O on any chord of the system, meeting the cubic in the points O_1, O_2, O_3, such that $OO_1 + OO_2 + OO_3 = 0$, is

$$(m \sin C - n \sin B)^2 \frac{d^2 u}{dx^2} + (n \sin A - l \sin C)^2 \frac{d^2 u}{dy^2} + (l \sin B - m \sin A)^2 \frac{d^2 u}{dz^2}$$
$$+ 2\,(n \sin A - l \sin C)\,(l \sin B - m \sin A)\, \frac{d^2 u}{dy\, dz}$$
$$+ 2\,(l \sin B - m \sin A)\,(m \sin C - n \sin B)\, \frac{d^2 u}{dz\, dx}$$
$$+ 2\,(m \sin C - n \sin B)\,(n \sin A - l \sin C)\, \frac{d^2 u}{dx\, dy} = 0;$$

and it is a question worth considering, what is the envelope of these diameters?

Writing, for shortness,

$$m \sin C - n \sin B = \lambda,$$
$$n \sin A - l \sin C = \mu,$$
$$l \sin B - m \sin A = \nu;$$

then, identically, $\lambda \sin A + \mu \sin B + \nu \sin C = 0.$

If, further, a, b, c, f, g, h be written for $\frac{d^2u}{dx^2} \ldots \frac{d^2u}{dx\,dy}$, then the equation to a diameter is

$$a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu = 0,$$

and the envelope of this line, where the parameters λ, μ, ν vary, subject to the above relation, is at once seen to be

$$(bc - f^2) \sin^2 A + (ca - g^2) \sin^2 B + (ab - h^2) \sin^2 C + 2\,(gh - af) \sin B \sin C + 2\,(hf - bg) \sin C \sin A + 2\,(fg - ch) \sin A \sin B = 0;$$

which, in relation to these "diameters," may be called the "central conic."

The preceding method may obviously be generalised. The equation to any "diametral curve" is, whatever the order of u,

$$\lambda^r \frac{d^r u}{dx^r} + \ldots + r\lambda^r \mu \frac{d^r u}{dx^{r-1} dy} + \ldots = 0,$$

and the envelope of this, subject to

$$\lambda \sin A + \mu \sin B + \nu \sin C = 0,$$

may similarly be sought.

The "diametral conics" of a cubic, in particular, pass through the four fixed points common to the polar conics of all points on the line at infinity.

On the Twenty-one Coordinates of a Conic in Space.

By William Spottiswoode, P.R.S.

[*Read June 12th*, 1879.]

In a Note published in the Report of the British Association for 1878 (Dublin Meeting), I gave a short account of the coordinates (then described as 18 in number) of a conic in space, and of the equations of condition by which they were connected. A modification of the notation there used will perhaps place these quantities in a clearer light.

The six coordinates of a right line may be derived from the equations of the two planes of which it is the intersection, by eliminating each of the variables in succession. Thus, if the equations of the two planes be

$$\alpha x + \beta y + \gamma z + \delta t = 0,$$
$$\alpha' x + \beta' y + \gamma' z + \delta' t = 0;$$

then, by eliminating t, we obtain

$$(\alpha\delta' - \alpha'\delta)\, x + (\beta\delta' - \beta'\delta)\, y + (\gamma\delta' - \gamma'\delta)\, z = 0.$$

Or, adopting the usual notation, we have the following expression for the six coordinates in question, a, b, c, f, g, h; viz.,

$$\mathrm{a, b, c, f, g, h} = \begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix};$$

the foregoing equation may be written thus

$$\mathrm{f}x + \mathrm{g}y + \mathrm{h}z = 0.$$

And all the four forms derived by the elimination of x, y, z, t, in turn, from the equations of the planes, may be comprised in the formula

$$\begin{pmatrix} . & \mathrm{c}, & -\mathrm{b}, & \mathrm{f} \\ -\mathrm{c}, & . & \mathrm{a}, & \mathrm{g} \\ \mathrm{b}, & -\mathrm{a}, & . & \mathrm{h} \\ \mathrm{f}, & \mathrm{g}, & \mathrm{h}, & . \end{pmatrix} (x, y, z, t) = 0.$$

And the six coordinates a, ..., f, ..., are, as is well-known, connected by the following relation $\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0$.

In like manner, if a conic in space be determined by the intersection of the quadric surface

$$(a, b, c, d, f, g, h, l, m, n)\,(x, y, z, t)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

by the plane $$\alpha x + \beta y + \gamma z + \delta t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

we may, by eliminating the variables in turn, obtain therefrom the following forms; viz.,

$$\left.\begin{array}{l} a(\beta y + \gamma z + \delta t)^2 - 2\alpha\,(\beta y + \gamma z + \delta t)(hy + gz + lt) + \alpha^2\,(b, c, d, n, m, f)(y, z, t)^2 = 0 \\ b\,(\alpha x + \gamma z + \delta t)^2 - 2\beta\,(\alpha x + \gamma z + \delta t)(hx + fz + mt) + \beta^2\,(a, c, d, n, l, g)(x, z, t)^2 = 0 \\ c\,(\alpha x + \beta y + \delta t)^2 - 2\gamma\,(\alpha x + \beta y + \delta t)(gx + fy + nt) + \gamma^2\,(a, b, d, m, l, h)(x, y, t)^2 = 0 \\ d(\alpha x + \beta y + \gamma z)^2 - 2\delta\,(\alpha x + \beta y + \gamma z)(lx + my + nz) + \delta^2\,(a, b, c, f, g, h)(x, y, z)^2 = 0 \end{array}\right\} \ldots (3)$$

If then, dropping for the moment, the coefficients a, b, ..., we put

	a	b	c	d	f	g	h	l	m	n
$A = (0, -\gamma, \beta, 0)^2$ =	...	γ^2	β^2	...	$-2\beta\gamma$	...	...	...	...	...
$B = (\gamma, 0, -\alpha, 0)^2$ =	γ^2	...	α^2	...	...	$-2\gamma\alpha$	...	...	...	...
$C = (-\beta, \alpha, 0, 0)^2$ =	β^2	α^2	...	...	...	...	$-2\alpha\beta$	...	...	...
$F = (-\delta, 0, 0, \alpha)^2$ =	δ^2	...	...	α^2	...	...	...	$-2\alpha\delta$	...	...
$G = (0, -\delta, 0, \beta)^2$ =	...	δ^2	...	β^2	...	...	...	...	$-2\beta\delta$	...
$H = (0, 0, -\delta, \gamma)^2$ =	...	...	δ^2	γ^2	...	...	...	...	...	$-2\gamma\delta$
$-A' = (\gamma, 0, -\alpha, 0)(-\beta, \alpha, 0, 0)$ =	$-\beta\gamma$	...	...	...	$-\alpha^2$	$\alpha\beta$	$\alpha\gamma$	...	...	...
$-B' = (-\beta, \alpha, 0, 0)(0, -\gamma, \beta, 0)$ =	...	$-\alpha\gamma$	...	...	$\alpha\beta$	$-\beta^2$	$\beta\gamma$	...	...	...
$-C' = (0, -\gamma, \beta, 0)(\gamma, 0, -\alpha, 0)$ =	...	...	$-\alpha\beta$	...	$\alpha\gamma$	$\beta\gamma$	$-\gamma^2$	...	...	...
$F' = (0, -\delta, 0, \beta)(0, 0, -\delta, \gamma)$ =	...	...	...	$\beta\gamma$	δ^2	...	...	...	$-\delta\gamma$	$-\delta\beta$
$G' = (0, 0, -\delta, \gamma)(-\delta, 0, 0, \alpha)$ =	...	...	...	$\gamma\alpha$	...	δ^2	...	$-\delta\gamma$	...	$-\delta\alpha$
$H' = (-\delta, 0, 0, \alpha)(0, -\delta, 0, \beta)$ =	...	...	...	$\alpha\beta$	...	...	δ^2	$-\delta\beta$	$-\delta\alpha$	...
$P = (0, -\gamma, \beta, 0)(-\delta, 0, 0, \alpha)$ =	...	...	...	...	...	$-\beta\delta$	$\gamma\delta$	...	$-\alpha\gamma$	$\alpha\beta$
$L = (-\beta, \alpha, 0, 0)(-\delta, 0, 0, \alpha)$ =	$\beta\delta$	...	...	...	...	...	$-\alpha\delta$	$-\alpha\beta$	α^2	...
$-L' = (\gamma, 0, -\alpha, 0)(-\delta, 0, 0, \alpha)$ =	$-\gamma\delta$	...	...	...	...	$\alpha\delta$	...	$\alpha\gamma$	...	$-\alpha^2$
$Q = (\gamma, 0, -\alpha, 0)(0, -\delta, 0, \beta)$ =	...	...	...	...	$\alpha\delta$	...	$-\gamma\delta$	$\beta\gamma$	...	$-\alpha\beta$
$M = (0, -\gamma, \beta, 0)(0, -\delta, 0, \beta)$ =	...	$\gamma\delta$	...	...	$-\beta\delta$	...	...	...	$-\beta\gamma$	β^2
$-M' = (-\beta, \alpha, 0, 0)(0, -\delta, 0, \beta)$ =	...	$-\alpha\delta$	...	...	...	...	$\beta\delta$	$-\beta^2$	$\alpha\beta$	...
$R = (-\beta, \alpha, 0, 0)(0, 0, -\delta, \gamma)$ =	...	...	...	...	$-\alpha\delta$	$\beta\delta$	...	$-\beta\gamma$	$\alpha\gamma$	...
$N = (\gamma, 0, -\alpha, 0)(0, 0, -\delta, \gamma)$ =	...	...	$\alpha\delta$	...	...	$-\gamma\delta$	...	γ^2	...	$-\alpha\gamma$
$-N = (0, -\gamma, \beta, 0)(0, 0, -\delta, \gamma)$ =	...	...	$-\beta\delta$	...	$\gamma\delta$	...	...	...	$-\gamma^2$	$\beta\gamma$

} ..... (4).

Then, the equations (3) take the forms

$$\left.\begin{aligned}(C, B, F, L, L', A')\,(y, z, t)^2 = 0 \text{ or } X = 0\\ (C, A, G, M, M', B')\,(x, z, t)^2 = 0 \text{ or } Y = 0\\ (B, A, H, N, N', C')\,(x, y, t)^2 = 0 \text{ or } Z = 0\\ (F, G, H, F', G', H')(x, y, z)^2 = 0 \text{ or } T = 0\end{aligned}\right\} \quad \ldots\ldots\ldots(5).$$

And the twenty-one quantities $A, B, \ldots$, may be called the "Twenty-one Coordinates of a Conic in Space." Or if, as in the Note quoted at the outset, we omit the three quantities P, Q, R, which do not occur in the forms 5, the remaining eighteen quantities may be called the "Eighteen Coordinates of a Conic in Space."

These coordinates, whether twenty-one or eighteen, like the six coordinates of a straight line, are not independent. In fact, by an inspection of their forms, it is not difficult to see that they satisfy the following equations; viz.,

$$\left.\begin{aligned}
. \quad -H\beta + F'\gamma + N'\delta &= 0\\
. \quad F'\beta - G\gamma + M\delta &= 0\\
. \quad N\beta + M\gamma - A\delta &= 0\\
-H\alpha \quad . \quad + G'\gamma + N\delta &= 0\\
G'\alpha \quad . \quad - F\gamma + L'\delta &= 0\\
N\alpha \quad . \quad + L'\gamma - B\delta &= 0\\
-G\alpha + H'\beta \quad . \quad + M'\delta &= 0\\
H'\alpha - F\beta \quad . \quad + L\delta &= 0\\
M'\alpha + L\beta \quad . \quad - C\delta &= 0\\
-A\alpha + C'\beta + B'\gamma \quad . \quad &= 0\\
C\alpha - B\beta + A'\gamma \quad . \quad &= 0\\
B'\alpha + A'\beta - C\gamma \quad . \quad &= 0
\end{aligned}\right\} \quad \ldots\ldots\ldots(6);$$

and, if from these we eliminate $\alpha, \beta, \gamma, \delta$, so far as they appear explicitly, we shall obtain the following relations between $A, B, \ldots$; viz.,

$$\begin{vmatrix}
. & . & . & -H, & G', & N, & -G, & H', & M', & -A, & C', & B'\\
-H, & F', & N', & . & . & . & H', & -F, & L, & C', & -B, & A\\
F', & -G, & M, & G', & -F, & L', & . & . & . & B', & A', & -C\\
N', & M, & -A, & N, & L', & -B, & M', & L, & -C, & . & . & .
\end{vmatrix} = 0 \quad \ldots\ldots\ldots(7).$$

These relations are nine in number; and as all are concerned with the ratios, and not the absolute values of the quantities in question, the total number of independent coordinates will be $18-9-1=8$, as it should be. As regards the three quantities P, Q, R, which do not appear in (6) or (7); we have, in the first place,

$$P + Q + R = 0 \quad \ldots\ldots\ldots(8);$$

secondly,

$$\begin{aligned} \alpha P + \beta L' - \gamma L &= 0, \\ -\alpha M + \beta Q + \gamma M' &= 0, \\ \alpha N' - \beta N + \gamma R &= 0; \end{aligned}$$

whence

$$\begin{vmatrix} P, & L', & -L \\ -M, & Q, & M' \\ N', & -N, & R \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9);$$

and the equations (8) and (9) express two relations between P, Q, R, and the other eighteen coordinates. To find the third relation, it will be convenient to remark that the twenty-one coordinates are the twenty-one minors of the matrix

$$\begin{matrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & a, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & . \end{matrix}$$

formed by selecting two out of the first four rows and columns, and adding to them the fifth row and column. Thus,

$$\begin{matrix} A = b, & f, & \beta, \\ f, & c, & \gamma, \\ \beta, & \gamma, & . \end{matrix}$$

and, more generally,

$$\left.\begin{matrix} h, b, f, m, \beta = -A, C', B', P, -M, N' \\ g, f, c, n, \gamma \\ \alpha, \beta, \gamma, \delta, . \\ a, h, g, l, \alpha = P, L', -L, -F, -H', -G' \\ l, m, n, d, \gamma \\ \alpha, \beta, \gamma, \delta, . \\ g, f, e, n, \gamma = C', -B, A', L', Q, -N \\ a, h, g, l, \alpha \\ \alpha, \beta, \gamma, \delta, . \\ h, b, f, m, \beta = -M, Q, M', -H', -G, -F' \\ l, m, n, d, \delta \\ \alpha, \beta, \gamma, \delta, . \\ a, h, g, l, \alpha = B', A', -C, -L, M', R \\ h, b, f, m, \beta \\ \alpha, \beta, \gamma, \delta, . \\ g, f, c, n, \gamma = N', -N, R, -G', -F', -H \\ l, m, n, d, \delta \\ \alpha, \beta, \gamma, \delta, . \end{matrix}\right\} \quad \ldots\ldots\ldots (10).$$

And, consequently, from the identity

$$\begin{array}{l} a,\ h,\ g,\ l,\ \alpha,\ \alpha \\ h,\ \beta,\ f,\ m,\ \beta,\ \beta \\ g,\ f,\ c,\ n,\ \gamma,\ \gamma \\ l,\ m,\ n,\ d,\ \delta,\ \delta \\ \alpha,\ \beta,\ \gamma,\ \delta,\ .\ . \\ \alpha,\ \beta,\ \gamma,\ \delta,\ .\ . \end{array}$$

we may obtain any of the three following forms; viz.,

$$\left.\begin{array}{rl} AF - B'G' - CH' + P^2 & - L'M - LN' = 0 \\ -A'F' + BG - CH' - ML' + Q^2 & - M'N = 0 \\ -A'F - B'G' + CH - NL - NM' + R^2 & = 0 \end{array}\right\} \quad \ldots\ldots(11),$$

any one of which may be taken as a third relation connecting P, Q, R with the other eighteen quantities.

The equation of the quadric (1) admits of a peculiar transformation by means of the coordinates of the conic in question, as follows:— Adding together the equations (5), we find

$$\begin{aligned} & X+Y+Z+T \\ = \ & (B+C+F)\,x^2+2\,(A'+F')\,yz+2\,(M'+N)\,xt \\ & +(C+A+G)\,y^2+2\,(B'+G')\,zx+2\,(N'+L)\,yt \\ & +(A+B+H)\,z^2+2\,(C'+H')\,xy+2\,(L'+M)\,zt \\ & +(F+G+H)\,t^2 \\ = \ & (A+B+C+F+G+H)\,(x^2+y^2+z^2+t^2) \\ & -Hy^2-Gz^2-At^2+2\,(Mzt+N'yt+F'yz) \\ & -Hx^2-Fz^2-Bt^2+2\,(L'zt+Nxt+G'zx) \\ & -Gx^2-Fy^2-Ct^2+2\,(Lyt+M'xt+H'xy) \\ & -Ax^2-By^2-Cz^2+2\,(A'yz+B'zx+C'xy). \end{aligned}$$

But, in virtue of the conditions (7), each of the expressions forming the last four lines may be resolved into the product of two linear factors, say, p, p'; q, q'; r, r'; s, s'. Hence, putting

$$\rho^2 = x^2+y^2+z^2+t^2,$$

$$\begin{aligned} S &= A+B+C+F+G+H \\ &= (a+b+c+d)(\alpha^2+\beta^2+\gamma^2+\delta^2) \\ &\qquad -(a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n)\,(\alpha,\ \beta,\ \gamma,\ \delta)^2, \end{aligned}$$

the equation in question takes the form

$$S\rho^2 - pp' - qq' - rr' - ss' = 0.$$

But, on the other hand,

$$(H, G, A, -M, -N', -F)(y, z, t)^2 = (b, c, d, m, n, f)(z\delta - t\gamma, t\beta - y\delta, y\gamma - z\beta)^2,$$

$$(H, F, B, -L', -N, -G)(x, z, t)^2 = (a, c, d, l, n, g)(z\delta - t\gamma, t\alpha - x\delta, x\gamma - z\alpha)^2,$$

$$(G, F, C, -L, -M', -H)(x, y, t)^2 = (a, b, d, m, l, h)(y\delta - t\beta, t\alpha - x\delta, x\beta - y\alpha)^2,$$

$$(A, B, C, -A', -B', -C')(x, y, z)^2 = (a, b, c, f, g, h)(y\gamma - z\beta, z\alpha - x\gamma, x\beta - y\alpha)^2.$$

And these expressions vanish respectively for the values

$$y : z : t = \beta : \gamma : \delta,$$
$$x : z : t = \alpha : \gamma : \delta,$$
$$x : y : t = \alpha : \beta : \delta,$$
$$x : y : z = \alpha : \beta : \gamma;$$

which are, consequently, the equations to the lines in which the pairs of planes p, p'; q, q'; r, r'; s, s', respectively intersect. It may also be added that the plane containing the two lines

$$q, q';\ r, r' \text{ is } \delta x - \alpha t = 0;$$
$$r, r';\ p, p' \text{ ,, } \delta y - \beta t = 0;$$
$$p, p';\ q, q' \text{ ,, } \delta z - \gamma t = 0;$$
$$p, p';\ s, s' \text{ ,, } \beta z - \gamma y = 0;$$
$$q, q';\ s, s' \text{ ,, } \gamma x - \alpha z = 0;$$
$$r, r';\ s, s' \text{ ,, } \alpha y - \beta x = 0.$$

The condition that two conics in space may meet.

The two conics may, in general, be considered as the sections of two quadric surfaces S, S_1, by two planes L, L_1. And if $S = 0$, $L = 0$ be the equations to the first conic, say the conic (S, L); and $S_1 = 0$, $L_1 = 0$ those of the second conic, the condition required will be found by eliminating the variables from the four equations $S = 0$, $L = 0$, $S_1 = 0$, $L_1 = 0$. This may be effected in more than one way. First, we may eliminate from the equations as they stand, but the result would not be in a form suited to the present purpose. Secondly, we may eliminate one of the variables, say t, from the equations S, L; the same from the equations S_1, L_1; and also from the equations L, L_1. We shall then have three equations in x, y, z; and the elimination will give us a relation between the coordinates of the conics (S, L), (S_1, L_1)

of the degree 2, and the coordinates of the line (L, L_1) of the degree 4. Thirdly, we may eliminate the variables from two of the equations (5), and two of the corresponding equations of the second conic. The results of these three methods are, of course, substantially the same, but they differ in form.

By the second method, we shall have to eliminate x, y, z from equations which may be written thus

$$(F,\ G,\ H,\ F',\ G',\ H')\,(x, y, z)^2 = 0,$$
$$(F_1,\ G_1,\ H_1,\ F_1',\ G_1',\ H_1')\,(x, y, z)^2 = 0,$$
$$\mathrm{f}x+\mathrm{g}y+\mathrm{h}z = 0.$$

Where $F_1, G_1, \ldots$ represent the coordinates of the conic (S_1, L_1). Multiplying the last of these three equations by x, y, z, respectively, we shall have five equations in $x^2, y^2, z^2, yz, zx, xy$. These give the following values

$$\begin{aligned} x^2 : y^2 : z^2 : yz : zx : xy = {} & F,\ G,\ H,\ F',\ G',\ H', \\ & F_1,\ G_1,\ H_1,\ F_1',\ G_1',\ H_1', \\ & \mathrm{f},\ .\ .\ .\ \mathrm{h},\ \mathrm{g}, \\ & .\ \mathrm{g},\ .\ \mathrm{h},\ .\ \mathrm{f}, \\ & .\ .\ \mathrm{h},\ \mathrm{g},\ \mathrm{f},\ . \end{aligned}$$

and if, for brevity, we write $FG_1 - F_1G = (F, G)$, &c., it will be found *à une facteur près*, that

$$\begin{aligned} \tfrac{1}{2}x^2 = {} & -(H, G')\,\mathrm{g}^3+(G, H')\,\mathrm{h}^3 \\ & +[2(G', H')+(H, H')]\,\mathrm{g}^2\mathrm{h}-(G, F')\,\mathrm{h}^2\mathrm{f} \quad . \\ & +[2(H', F')-(G, G')]\,\mathrm{gh}^2 \quad . \quad +(H, F')\,\mathrm{fg}^2+(G, H)\,\mathrm{fgh}, \\ \tfrac{1}{2}y^2 = {} & -(F, H')\,\mathrm{h}^3+(H, F')\,\mathrm{f}^3 \\ & + \quad . \quad +[2(G', H')+(F\ F')]\,\mathrm{h}^2\mathrm{f}-(H, G')\,\mathrm{f}^2\mathrm{g} \\ & +(F, G')\,\mathrm{gh}^2+[2(F', G')-(H, H')]\,\mathrm{hf}^2 \quad . \quad +(H, F)\,\mathrm{fgh}, \\ \tfrac{1}{2}z^2 = {} & -(G, F')\,\mathrm{f}^3+(F, G')\,\mathrm{g}^3 \\ & -(F, H')\,\mathrm{g}^2\mathrm{h} \quad . \quad +[2(H', F')+(G, G')]\,\mathrm{f}^2\mathrm{g} \\ & . \quad + G, H')\,\mathrm{f}^2\mathrm{h}+[2(G', H')-(F, F')]\,\mathrm{fg}^2+(F, G)\,\mathrm{fgh}, \end{aligned}$$

$$\begin{aligned} yz = {} & (G, H)\,\mathrm{f}^3-2(F, G')\,\mathrm{g}^2\mathrm{h}-2(F, G)\,\mathrm{h}^2\mathrm{f}+2(H, H')\,\mathrm{f}^2\mathrm{g} \\ & +2(F, H')\,\mathrm{gh}^2-2(G, G')\,\mathrm{hf}^2-(H, F)\,\mathrm{fg}^2-4(G', H')\,\mathrm{fgh}, \\ zx = {} & (H, F)\,\mathrm{g}^3+2(F, F')\,\mathrm{g}^2\mathrm{h}-2(G, H')\,\mathrm{h}^2\mathrm{f}-(G, H)\,\mathrm{f}^2\mathrm{g} \\ & -(F, G)\,\mathrm{gh}^2+2(G, F')\,\mathrm{hf}^2-2(H, H')\,\mathrm{fg}^2-4(H', F')\,\mathrm{fgh}, \\ xy = {} & (F, G)\,\mathrm{h}^3-(H, F)\,\mathrm{g}^2\mathrm{h}+2(G, G')\,\mathrm{h}^2\mathrm{f}-2(H, F')\,\mathrm{f}^2\mathrm{g} \\ & -(F, F')\,\mathrm{gh}^2-(G, H)\,\mathrm{hf}^2+2(H, G')\,\mathrm{fg}^2-4(F', G')\,\mathrm{fgh}. \end{aligned}$$

If, from these equations, we form the identical expression

$$y^2 . z^2 - (yz)^2 = 0,$$

it will be easily seen that the term independent of f will vanish. The coefficient of f in the expression for $y^2 . z^2$ will be

$$4\{(FG')\,g-(FH')\,h\}\{(HF)\,g^2+4(G'H')\,gh+(FG)\,h^2\}\,gh,$$

while that in the expression for $(yz)^2$ will be the same. Hence the whole expression will be divisible by f^2. And, after various reductions, the final result will be found to be

$$\begin{aligned}
&-f^4\{(GH)^2+4(GF')(HF')\}\\
&-g^4\{(HF)^2+4(HG')(FG')\}\\
&-h^4\{(FG)^2+4(FH')(GH')\}\\
&+4g^3h\{-(HF)(FF')+(HG')(FH')+(FG')(HH')+4(FG')(F'G')\}\\
&+4gh^3\{\ (FG)(FF')+(FG')(GH')+(FH')(GG')-4(FH')(H'F')\}\\
&+4h^3f\{-(FG)(GG')+(FH')(GF')+(FF')(GH')+4(GH')(G'H')\}\\
&+4hf^3\{\ (GH)(GG')+(GH')(HF')+(GF')(HH')-4(GF')(F'G')\}\\
&+4f^3g\{-(GH)(HH')+(GF')(HG')+(GG')(HF')+4(HF')(H'F')\}\\
&+4fg^3\{\ (HF)(HH')+(HF')(FG')+(FF')(HG')-4(HG')(G'H')\}\\
&+2g^2h^2\{(HF)(FG)-2(FF')^2-2(FG')(GG')-2(FH')(HH')\\
&\qquad\qquad+4(FG')(H'F')-4(FH')(F'G')\}\\
&+2h^2f^2\{(FG)(GH)-2(GG')^2-2(GH')(HH')-2(GF')(FF')\\
&\qquad\qquad+4(GH')(F'G')-4(GF')(G'H')\}\\
&+2f^2g^2\{(GH)(HF)-2(HH')^2-2(HF')(FF')-2(HG')(GG')\\
&\qquad\qquad+4(HF')(G'H')-4(HG')(H'F')\}\\
&+4f^2gh\{2(GH)(G'H')-(HF)(GF')+(FG)(HF')\\
&\qquad\qquad+(GG')(HH')-(GH')(HG')\\
&\qquad\qquad+2(GG')(F'G')-2(HH')(H'F')+4(H'F')(F'G)\}\\
&+4fg^2h\{(GH)(FG')+2(HF)(H'F')-(FG)(HG')\\
&\qquad\qquad+(HH')(FF')-(HF')(FH')\\
&\qquad\qquad+2(HH)(G'H')-2(FF')(F'G')+4(G'H')(F'G')\}\\
&+4fgh^2\{-(GH)(FH')+(HF)(GH')+2(FG)(F'G')\\
&\qquad\qquad+(FF')(GG')-(FG')(GF')\\
&\qquad\qquad+2(FF')(H'F)-2(GG')(G'H')+4(H'F')(G'H')\}.
\end{aligned}$$

The results due to eliminating the variables from the other forms of (5) will be obtained by writing

in turn	c, −b, f; and	C, B, F, L, L', A';
	−c, a, g;	C, A, G, M, M', B';
	b, −a, h;	B, A, H, N, N', C';
	f, g, h;	F, G, H, F', G', H'.

For the third method we may proceed as follows. If $X_1=0$, $Y_1=0$, $Z_1=0$, $T_1=0$, be the equations corresponding to (5) for the second conic, we may eliminate z from the two equations $Y=0$, $Y_1=0$; and y from the two equations $Z=0$, $Z_1=0$: these will give two equations of the degree 4 in $x : t$, and of the degree 2 in the coordinates of each of the conics. The elimination of $x : t$ from these two equations will give a resultant of the degree $4\times 2\times 2 = 16$ in the coordinates of each of the conics. But it seems doubtful whether it would be worth while actually to effect the elimination.

Addition by Prof. Cayley.

Write $U = (a, b, c, d, f, g, h, l, m, n\,\mathcal{X}\,x, y, z, t)^2$,

$U_0 = (\quad\quad\quad

And if we expand first in ξ, η, ζ, ω, and then in x, y, z, t, the final result is

	x^2	y^2	z^2	t^2	yz	zx	xy	xt	yt	zt	
ξ^2		C	B	F	$2A'$				$2L$	$2L'$	$= 0.$
$+\ \eta^2$	C		A	G		$2B'$		$2M'$		$2M$	
$+\ \zeta^2$	B	A		H			$2C'$	$2N$	$2N'$		
$+\ \omega^2$	F	G	H		$2F'$	$2G'$	$2H'$				
$+\eta\zeta$	$2A'$			$2F'$	$-2A$	$-2C'$	$-2B'$	$2\,(Q-R)$	$-2M$	$-2N'$	
$+\zeta\xi$	.	$2B'$		$2G'$	$-2C'$	$-2B$	$-2A'$	$-2L'$	$2\,(R-P)$	$-2N$	
$+\xi\eta$			$2C'$	$2H'$	$-2B'$	$-2A'$	$-2C$	$-2L$	$-2M'$	$2\,(P-Q)$	
$+\xi\omega$		$2M'$	$2N$		$2\,(Q-R)$	$-2L'$	$-2L$	$-2F$	$-2H'$	$-2G'$	
$+\eta\omega$	$2L$		$2N'$		$-2M$	$-2\,(R-P)$	$-2M'$	$-2H'$	$-2G$	$-2F'$	
$+\zeta\omega$	$2L'$	$2M$			$-2N'$	$-2N$	$2\,(P-Q)$	$-2G'$	$-2F'$	$-2H$	

In particular, if $\eta=0$, $\zeta=0$, $\omega=0$, then we have the foregoing equation $X=0$; and the like for the equations $Y=0$, $Z=0$, and $W=0$ respectively.

Take a, b, c, f, g, h for the six coordinates of the line through the point

$$\left|\begin{matrix} x, & y, & z, & t \\ \xi, & \eta, & \zeta, & \omega \end{matrix}\right|$$

that is, write

$$\begin{aligned} \mathrm{a} &= y\zeta - z\eta, & \mathrm{f} &= x\omega - t\xi, \\ \mathrm{b} &= z\xi - x\eta, & \mathrm{g} &= y\omega - t\eta, \\ \mathrm{c} &= x\eta - y\xi, & \mathrm{h} &= z\omega - t\zeta, \end{aligned}$$

where, of course,

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch} = 0.$$

Then the foregoing equation of the cone is

$$\left.\begin{aligned} &A\mathrm{a}^2 + B\mathrm{b}^2 + C\mathrm{c}^2 + F\mathrm{f}^2 + G\mathrm{g}^2 + H\mathrm{h}^2 \\ &-2A'\mathrm{bc} - 2B'\mathrm{ca} - 2C'\mathrm{ab} + 2F'\mathrm{gh} + 2G'\mathrm{hf} + 2H'\mathrm{fg} \\ &+2P\mathrm{af} + 2M\mathrm{ag} - 2N'\mathrm{ah} \\ &-2L'\mathrm{bf} + 2Q\mathrm{bg} + 2N\mathrm{bh} \\ &+2L\mathrm{cf} - 2M'\mathrm{cg} + 2R\mathrm{ch} \end{aligned}\right\} = 0.$$

And this may be regarded as the equation of the *conic* in terms of the twenty-one coordinates of the conic, and of the six coordinates of an arbitrary line meeting the conic. It is, in fact, the general form of the equation given in the paper—Cayley "On a new Analytical Representation of a Curve in Space," Quart. Math. Jour., t. iii. (1860), see p. 233.

Déduction de différents Théorèmes géométriques d'un seul Principe Algébrique. *Par* H. G. Zeuthen.*

Le principe algébrique dont nous allons exposer ici plusieurs applications géométriques s'exprime par le Théorème suivant :

Soit donnée une forme algébrique entière, homogène et du second degré par rapport à chacune de deux couples de variables x_1, x_2 et ξ_1, ξ_2 ; formons les deux discriminants de cette forme, qui seront des formes quartiques contenant la couple des variables regardées comme constantes pendant la formation du discriminant : je dis que ces deux discriminants auront les mêmes invariants.

J'ai déduit ce théorème général d'une de ses applications géomé-

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* Received from the Author too late for communication at the Meeting on June 12.

triques, qui est bien connue (la 2me) ; mais il sera juste d'en communiquer ici une démonstration algébrique directe, que je dois à un de mes élèves, Mr. E. Schmidt, à qui j'avais communiqué le théorème. On y fait usage du calcul symbolique de *Aronhold* et *Clebsch*.*

Soient $a_x^2 a_\xi^2 = b_x^2 \beta_\xi^2 = c_x^2 \gamma_\xi^2 = d_x^2 \delta_\xi^2 = e_x^2 \epsilon_\xi^2 = f_x^2 \phi_\xi^2$

des représentations symboliques équivalentes de la forme donnée, et

$$A_x^4 = B_x^4 = C_x^4 = (\alpha\beta)^2 a_x^2 b_x^2 = (\gamma\delta)^2 c_x^2 d_x^2 = (\epsilon\phi)^2 e_x^2 f_x^2 \ldots \quad \ldots(1)$$

des représentations de son discriminant par rapport aux variables ξ. On en déduit, par formation successive de polaires, et par des permutations de symboles équivalents,

$$4A_x^3 A_y = (\alpha\beta)^2 (2a_x a_y b_x^2 + 2a_x^2 b_x b_y) = 4 (\alpha\beta)^2 a_x a_y b_x^2,$$

$$3A_x^2 A_y^2 = (\alpha\beta)^2 (a_y^2 b_x^2 + 2a_x a_y b_x b_y),$$

ou bien, à cause de l'identité

$$2a_x a_y b_x b_y = a_x^2 b_y^2 + a_y^2 b_x^2 - (ab)^2 (xy)^2,$$

$$3A_x^2 A_y^2 = (\alpha\beta)^2 [3a_x^2 b_y^2 - (ab)^2 (xy)^2] \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

La forme $A_x^4 = B_x^4 = C_x^4$ a les deux invariants indépendents entre eux :

$$I = (AB)^4, \quad J = (AB)^2 (BC)^2 (CA)^2.$$

Les formules (1) et (2), où l'on peut substituer des symboles (B_2 et $-B_1$; a_2 et $-a_1$, etc.) aux variables, nous permettront d'exprimer I et J par les symboles de la forme donnée. On aura

$$I = (AB)^4 = (\gamma\delta)^2 (Ac)^2 (Ad)^2$$
$$= \tfrac{1}{3} (\alpha\beta)^2 (\gamma\delta)^2 [3 (ac)^2 (bd)^2 - (ab)^2 (cd)^2] ;$$

et, en posant

$$K = (\alpha\beta)^2 (\gamma\delta)^2 (ac)^2 (bd)^2 = (a\gamma)^2 (\beta\delta)^2 (ab)^2 (cd)^2 \ldots\ldots\ldots(3),$$

$$L = (\alpha\beta)^2 (ab)^2 = (\gamma\delta)^2 (cd)^2 = (\epsilon\phi)^2 (ef)^2 \ldots\ldots\ldots\ldots\ldots(4),$$

on trouve $$I = K - \tfrac{1}{3} L^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Les invariants K et L dépendent d'une manière symétrique des symboles grecs et romains ; l'invariant I appartient donc aussi au discriminant pris par rapport aux variables x.

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* Voir par exemple *Clebsch-Lindemann :* Vorlesungen über Geometrie, T. I., 3me section. On reconnaîtra, au point de départ de la démonstration actuelle, les opérations dont se sert Clebsch pour former une expression de $H_x^2 H_y^2$.

On trouve de même, par la formule (2),

$$\begin{aligned} 3J &= 3\,(AB)^2\,(BC)^2\,(CA)^2 \\ &= 3\,(BC)^2\,(a\beta)^2\,(aB)^2\,(bC)^2 - (a\beta)^2\,(ab)^2\,(BC)^4 \\ &= 3\,(a\beta)^2\,(aB)^2\,(bC)^2\,(BC)^2 - LI \\ &= 3\,(a\beta)^2\,(aB)^2\,(bC)^2\,(BC)^2 - KL + \tfrac{1}{3}L^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6). \end{aligned}$$

Le premier terme de cette expression se transforme de la manière suivante,

$$\begin{aligned} 3\,(a\beta)^2\,(aB)^2\,(bC)^2\,(BC)^2 &= (a\beta)^2\,(bC)^2\,(\gamma\delta)^2\,[3\,(ca)^2\,(dC)^2 - (cd)^2\,(aC)^2] \\ &= 3\,(a\beta)^2\,(\gamma\delta)^2\,(ca)^2\,(bC)^2\,(dC)^2 - L\,(AC)^4 \\ &= 3\,(a\beta)^2\,(\gamma\delta)^2\,(ca)^2\,(bC)^2\,(dC)^2 - KL + \tfrac{1}{3}L^3 \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots (7). \end{aligned}$$

Nous transformerons encore le premier terme de cette dernière expression,

$$\begin{aligned} &3\,(a\beta)^2\,(\gamma\delta)^2\,(ca)^2\,(bC)^2\,(dC)^2 \\ &\quad = 3\,(a\beta)^2\,(\gamma\delta)^2\,(ca)^2\,(e\phi)^2\,(eb)^2\,(fd)^2 - (a\beta)^2\,(\gamma\delta)^2\,(ca)^2\,(e\phi)^2\,(ef)^2\,(bd)^2. \end{aligned}$$

Le dernier terme est égal à $-KL$; le premier terme est un invariant $3M$, qui prend, par inversion 1° des symboles $b\beta$ et $c\gamma$, 2° des symboles $d\delta$ et $e\epsilon$ de la forme donnée, la même forme qui résulterait de la substitution de symboles grecs à des symboles romains, et réciproquement, de façon que

$$\begin{aligned} M &= (a\beta)^2\,(\gamma\delta)^2\,(\epsilon\phi)^2\,(ac)^2\,(be)^2\,(df)^2 \\ &= (a\gamma)^2\,(\beta\epsilon)^2\,(\delta\phi)^2\,(ab)^2\,(cd)^2\,(ef)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8). \end{aligned}$$

Des formules (6), (7) et (8) résulte, par des substitutions successives,

$$J = M - KL + \tfrac{2}{9}L^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

La formation de J présente donc la même symétrie par rapport aux symboles grecs et romains que celle de I. Le théorème est donc démontré.

Si $x_1 : x_2$ et $\xi_1 : \xi_2$ sont les paramètres qui déterminent *les éléments x et ξ de deux séries unicursales* (de points d'une courbe, de tangentes à une courbe, etc.), on aura, en égalant la forme donnée à zéro, une relation qui fait correspondre à chaque point x deux points ξ, et à chaque point ξ deux points x; réciproquement, si l'on sait qu'une relation de cette espèce se fait exprimer algébriquement, on saura aussi que l'équation qui l'exprime a la forme indiquée. En égalant un des discriminants à zéro on déterminera les quatre éléments x (ou ξ) dont les éléments correspondants ξ (ou x) coïncident. Les deux discriminants ayant le même invariant absolu $\frac{I^3}{J^2}$, on voit que *les* 6 *rapports an-*

harmoniques des quatre éléments x auxquels correspondent des éléments coïncidents ξ, sont égaux à ceux des quatre éléments ξ auxquels correspondent des éléments coïncidents x. Nous allons en indiquer plusieurs applications.

1. Soient les éléments x les tangentes d'une conique plane, et soient les éléments ξ correspondant à une tangente x ses points d'intersection avec une autre conique dans le même plan. Les deux points ξ, correspondant à une tangente x, coïncident si x est tangente commune aux deux coniques, et les deux tangentes x correspondant à un point ξ coïncident, si ξ est point d'intersection des deux coniques. On voit donc que les tangentes communes à deux coniques ont, sur l'une de ces coniques, les mêmes rapports anharmoniques que les quatre points d'intersection ont sur l'autre.

2. Soient les éléments correspondants x et ξ les droites joignant deux points fixes d'une courbe plane du troisième ordre à un point mobile de la même courbe. La correspondance demandée a évidemment lieu entre les droites des deux faisceaux. On obtient ainsi le théorème connu, que les 6 rapports anharmoniques des quatre tangentes qu'on peut mener d'un point P d'une courbe du troisième ordre à cette courbe sont indépendents de la position de P sur la courbe.

La correspondance ayant encore lieu, si les droites correspondantes des faisceaux joignent deux points doubles d'une courbe du quatrième ordre à un point variable de la même courbe, on voit que les groupes de quatre tangentes qu'on peut mener à une courbe de quatrième ordre à deux points doubles de ces deux points ont les mêmes rapports anharmoniques. Ce théorème résulte aussi, en cas particulier, de l'application suivante.

3. Soient les éléments x des génératrices d'une des générations d'une surface du second ordre, et les éléments correspondants ξ les génératrices d'une des générations d'une autre surface qui rencontrent les droites x. La correspondance demandée a lieu, et l'on voit que : dans un faisceau de surfaces du second ordre, tous les groupes de 4 génératrices de la même génération d'une surface du faisceau qui sont tangentes à la courbe d'intersection ont les mêmes rapports anharmoniques.

En projetant la courbe d'intersection d'un point quelconque sur un plan, on obtient une courbe du quatrième ordre à deux points doubles, et qui n'est soumise à aucune autre particularité. Les contours apparents des surfaces du faisceau forment un système de coniques tangentes quatre fois à la courbe. On voit que les 8 tangentes communes à la courbe et à une conique de ce système forment deux groupes de quatre, et que tous les groupes de tangentes communes qu'on obtient

en considérant toutes les coniques ont, sur les coniques respectives, les mêmes rapports anharmoniques.*

4. Soit x une tangente variable à une courbe du quatrième ordre et de la quatrième classe, et soient les éléments correspondants ξ les deux points d'intersection de x avec la courbe. Selon les formules Plückeriennes la courbe aura, si elle n'est pas composée (voir l'application 1), un point double et deux points cuspidaux, une tangente double et deux tangentes d'inflexion.

La correspondance demandée a lieu ici. Les deux points ξ qui correspondent à une tangente x coïncident, 1° si x est la tangente double, et 2° si x est la tangente menée d'un des points cuspidaux (mais à un autre point de contact). Les quatre racines $x_1 : x_2$ de l'un des discriminants sont donc—1° les deux paramètres qui déterminent la tangente double (ou ses deux points de contact), 2° les paramètres qui déterminent les tangentes menées des deux points cuspidaux. Les quatres racines $\xi_1 : \xi_2$ de l'autre discriminant déterminent, selon le principe de dualité, 1° le point double (ou les deux tangentes en ce point), 2° les points d'intersection des tangentes d'inflexion avec la courbe.

Ici, où les deux discriminants se décomposent en deux facteurs rationnels, se présente encore la question sur l'ordre des éléments x et ξ, déterminés par les discriminants, qui ont des rapports anharmoniques égaux. Faut-il, en effet, faire correspondre aux deux paramètres x qui déterminent la tangente double les deux paramètres ξ du point double, ou ceux des points d'intersection des tangentes d'inflexion, ou un paramètre de chacun des deux couples? A cause de l'identité des rapports anharmoniques

$$(\xi'\xi''\xi'''\xi^{iv}) = (\xi''\xi'\xi^{iv}\xi''') = (\xi'''\xi^{iv}\xi'\xi'') = (\xi^{iv}\xi'''\xi''\xi'),$$

les deux premières hypothèses ne diffèrent pas entre elles. Pour décider si l'une ou l'autre des deux hypothèses différentes qui restent encore possibles a lieu, il suffit de considérer un cas particulier, par exemple celui où les deux points de contact de la tangente double, et en même temps les deux tangentes d'inflexion, coïncident.† Alors, si x' et x'' représentent la tangente double, ξ''' et ξ^{iv} les points d'intersection des tangentes d'inflexion, on aura $(x'x''x'''x^{iv}) = 1 = (\xi'\xi''\xi'''\xi^{iv})$,

* Nous avons démontré autrement le même théorème dans un mémoire sur les courbes du quatrième ordre à deux points doubles, qui est sous presse pour être inséré dans le bulletin ("Oversigter") de l'Académie Royale Danoise, 1879.

† Ce cas est possible sans qu'en même temps d'autres des éléments x, ou des éléments ξ, que nous considérons, coïncident. Voir la dernière page de mon mémoire sur les *systèmes de courbes planes* (Mémoires de l'Académie Royale Danoise 1873), où je détermine les nombres de courbes ayant la propriété indiquée ici (ou plutôt la propriété réciproque), et satisfaisant à des conditions données.

pendant que tous les rapports anharmoniques où ξ''' et ξ^{iv} ne composent pas la première ou la dernière couple du rapport anharmonique des ξ sont égaux à zéro ou à l'infini. La dernière hypothèse est donc impossible, et la première doit être juste.

Nous avons donc démontré la propriété suivante des courbes planes du quatrième ordre et de la quatrième classe ; les deux* rapports anharmoniques des deux points de contact de la tangente double et des deux points de contact des tangentes menées des points cuspidaux, sont égaux à ceux des deux tangentes au point double et des tangentes aux points d'intersection des tangentes d'inflexion. (Dans cet énoncé nous avons, pour plus de commodité, substitué les points de contact aux tangentes, et réciproquement.)

5. Considérons ensuite une courbe gauche caractérisée par les nombres Cayleyens suivants†

$$m = n = 5, \quad r = 6, \quad h = g = 4,$$
$$x = y = 5, \quad \alpha = \beta = 2.$$

On déduit de formules connues que la même courbe a 2 tangentes qui rencontrent encore une fois la courbe, et 2 plans osculateurs qui y sont encore une fois tangentes. Ces deux nombres résultent du reste aussi de l'application actuelle de notre principe.

Nous prenons pour éléments x les plans osculateurs à cette courbe unicursale. Chacun de ces plans a encore deux points d'intersection avec la courbe, que nous prenons pour éléments ξ correspondant au plan x. Réciproquement, par chaque point ξ passent 2 plans x osculateurs en des points différents de ξ. La correspondance demandée a donc lieu. Le premier discriminant de l'équation qui exprime cette correspondance détermine—1° les 2 plans osculateurs qui sont encore une fois tangentes, 2° les 2 plans qui passent par un point cuspidal ou stationnaire, et sont osculateurs en un autre point.‡ L'autre discriminant détermine—1° les 2 points de la courbe qui se trouvent sur des tangentes en d'autres points, 2° les 2 points d'intersection de la courbe avec les plans stationnaires.

Dans le cas particulier où un plan rencontre la courbe en cinq points consécutifs, les deux plans stationnaires—et par conséquent aussi leurs points d'intersection—et les deux plans osculateurs qui sont encore une fois tangents se confondent. On se sert—comme dans l'application 4—de

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* L'ordre des deux couples, ainsi que celui des éléments des couples, étant inconnu, le rapport anharmonique a ici deux valeurs différentes.

† Nous les désignons par les mêmes lettres que M. Salmon dans la "Geometry of three Dimensions."

‡ Par chacun des $\beta = 2$ points cuspidaux passe un seul de ces plans (Geometry of three Dimensions 3^{m} éd., p. 302.)

ce cas particulier pour déterminer les ordres des éléments x et ξ qui ont le même rapport anharmonique. On trouve ainsi la propriété suivante de la courbe : les deux rapports anharmoniques des 2 plans osculateurs qui sont encore une fois tangents et des 2 plans osculateurs qui passent par les points stationnaires (sans être osculateurs en ces points), sont égaux à ceux des 2 points où passent des tangentes en d'autres points et des 2 points d'intersection des plans stationnaires.

6. Soit donnée une surface du quatrième ordre à une droite double, et prenons pour éléments x les points de cette droite, pour éléments ξ correspondant à un point x, les plans tangents à la surface en x. La correspondance demandée a lieu alors entre les points x de la droite, et les plans ξ, qui forment un faisceau ayant la même droite pour axe. Les deux plans ξ tangents en un point x coïncident, si x est un point-pince (*pinch-point*) de la droite double ; les deux points de contact x d'un plan ξ coïncident, si ξ est un plan stationnaire de la droite double, c'est à dire un plan ayant sur la droite un contact qui le fait appartenir aux (β') plans tangents stationnaires de l'enveloppe des plans tangents stationnaires de la surface.* On voit donc que les quatre points-pinces sur une droite double d'une surface du quatrième ordre ont les mêmes rapports anharmoniques que les quatre plans stationnaires de la droite.

Si la surface a deux droites doubles, elle sera réglée, et les plans stationnaires de l'une des droites passeront par les points-pinces de l'autre. On voit donc que les points-pinces de l'une des droites ont les mêmes rapports anharmoniques que ceux de l'autre.

7. Une généralisation du dernier cas dont nous avons traité dans le No. 6 se présente presque immédiatement. Soient x et ξ des points mobiles de deux courbes unicursales dans l'espace, C_m et C_μ, des ordres m et μ, et soit donnée une relation entre les paramètres de x et ξ qui fait correspondre à chaque point x deux points ξ, et à chaque point ξ deux points x. Alors les droites x, ξ engendreront une surface gauche de l'ordre $2(m+\mu)$† et ayant les courbes C_m et C_μ pour courbes doubles. Notre principe nous montre que chacune de ces courbes a 4 points-pinces, et que les deux groupes de points-pinces ont les mêmes rapports anharmoniques.

Si les deux courbes sont planes et se trouvent dans le même plan, l'enveloppe des droites x, ξ sera une courbe de la classe $2(m+\mu)$. Les points-pinces seront alors remplacés par des points de cette enveloppe, qui ne seront pas en général ses seuls points d'intersection avec les courbes C_m et C_μ, mais qui seront déterminés par des facteurs rationnels

* Voir mon article : Sur les droites multiples des surfaces (Mathematische Annalen IV., p. 8.)

† On trouve, en effet, par le principe de correspondance, ce nombre de points d'intersection avec une droite quelconque.

du quatrième degré des premiers membres des équations servant à déterminer ces points d'intersection.

Il serait facile de multiplier ces applications,—nous proposons par exemple la discussion de la génération d'une courbe plane du troisième ordre comme lieu des points de contact des coniques d'un faisceau avec les droites d'un faisceau ;—mais nous croyons avoir montré déjà suffisamment l'unité de propriétés de figures très-différentes qu'établit le principe algébrique.

Note by Prof. Cayley.

The algebraical principle employed by Dr. Zeuthen, and from which he has deduced such interesting results, is certainly a well-known one. It is, for instance, assumed in my paper "On the Porism of the in-and-circumscribed Polygon, and on the (2, 2) Correspondence of the Points on a Conic," Quart. Math. Jour., t. xi. (1871), pp. 83—91, see p. 84, where I say that the two quartic functions are linearly transformable the one into the other. The two functions may be written

$$(ax^2+2bx+c)(a''x^2+2b''x+c'')-(a'x^2+2b'x+c')^2=(\mathrm{a, b, c, d, e}\mathcal{X}x,1)^4;$$

and

$$(ay^2+2a'y+a'')(cy^2+2c'y+c'')-(by^2+2b'y+b'')^2=(A,B,C,D,E\mathcal{X}y,1)^4;$$

the coefficients consequently are

$\mathrm{a} = aa''$	$-a'^2$	$A = ac$	$-b^2$
$4\mathrm{b} = 2ab''+2a''b$	$-4a'b'$	$4B = 2ac'+2a'c$	$-4bb'$
$6\mathrm{c} = ac''+a''c+4bb''$	$-2a'c'-4b'^2$	$6C = ac''+a''c+4a'c'$	$-2bb''-4b'^2$
$4\mathrm{d} = 2bc''+2b''c$	$-4b'c'$	$4D = 2a'c''+2a''c'$	$-4b'b''$
$\mathrm{e} = cc'$	$-c'^2$	$E = a''c''-b''^2.$	

I was under the impression that I had actually calculated the invariants of the two functions. For the invariant I, it can be done without difficulty ; the two forms of $12I$ are

$$\begin{aligned} &12(aa''-a'^2)(cc''-c'^2)\\ &-12(ab''+a''b-2a'b')(bc''+b''c-2b'c')\\ &+\quad(ac''+a''c+4bb''-2a'c'-4b'^2)^2, \end{aligned}$$

and

$$\begin{aligned} &12(ac-b^2)(a''c''-b''^2)\\ &-12(ac'+a'c-2bb')(a'c''+a''c'-2b'b'')\\ &+\quad(ac''+a''c+4a'c'-2bb''-4b'^2)^2, \end{aligned}$$

each of which is, in fact,

$$= a^2c''^2 + 14aca''c'' + a''^2c^2 - 12a'^2cc'' - 4a'c'\,(ac'' + a''c) - 12c'^2aa'' + 16a'^2c'^2$$
$$-12a''c''b^2 - (ac'' + a''c + 4a'c')\,(4bb'' + 8b'^2) - 12acb''^2$$
$$+24\,(a'c'' + a''c')\,bb' + 24\,(ac' + a'c)\,b'b'' + 16\,(bb'' - b'^2)^2.$$

The actual calculation of the other invariant J would be somewhat longer.

On Clifford's Graphs. *By* W. Spottiswoode, P.R.S.

[*Read June 12th*, 1879.]

In a very original paper, "On an Application of the new Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics," published in the "American Journal of Pure and Applied Mathematics," Vol. I., p. 64, Professor Sylvester states that he had "long been with a feeling of affinity, if not identity of object, between the inquiry into compound [chemical] radicals, and the search for 'Grundformen,' or irreducible invariants;" and that he was "agreeably surprised to find, of a sudden, distinctly pictured on my mental retina, a chemico-graphical image, serving to embody and illustrate the relations of those derived algebraical forms to their primitives, and to each other, which would perfectly accomplish the object which I had in view." "The factors of any algebraical form may be regarded as in some sense the analogues of the rays of atomicity in the equivalent chemical atom; these rays being what Dr. Frankland, according to his nomenclature, would have to designate as *free bonds*. Such rays between two consecutive atoms in a molecule are conceived as blending in some manner, so as to represent some unknown kind of special relation existing between them; they may then, with propriety, be called bonds, or lines of connection. An invariant of a form, or system of algebraical forms, must thus represent a saturated system of atoms, in which the rays of all the atoms are connected into bonds. Thus, *e.g.*, O_2 (oxygen combined with itself) will represent a quadric invariant of a quadric. Its graph is ○=○. Potash, a combination of potassium, oxygen, and hydrogen, having for its graph ⋀, will represent the invariant to a system of one quadratic and two linear forms. And, in general, the Jacobian to any two quantics will be completely expressed by their two corresponding atoms connected by a pair of bonds."

At the close of the paper, Prof. Sylvester adds, "The subjoined

matter is so exceedingly interesting, and throws such a flood of light on the chemico-algebraical theory," that he decides to publish it. The matter in question is a letter from Prof. Clifford, written at Gibraltar; and it contains the only general statement of his views on the subject ever published. The following is the passage which particularly refers to Graphs :—

"Another [paper] was to be about the very thing you speak of, which was communicated to the British Association at Bristol. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories, published ages ago in the 'Cambridge Mathematical Journal.' I have a strong impression that you will there find the analogy of covariants and invariants to compound radicals and saturated molecules.

"I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k=2$, and that I have altogether 6 sets of two variables each, namely

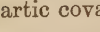
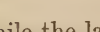
$$x_1x_2,\ y_1y_2,\ z_1z_2,\ u_1u_2,\ v_1v_2,\ w_1w_2.$$

Suppose the forms are

$$(xyzu),\ (yzvw),\ (xv),\ (uw)\ ;$$

viz., $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so on for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is : to obtain this invariant, regard the variables as alternate numbers, and simply multiply all the forms together. By alternate numbers, I mean those whose multiplication is polar ($xy = -yx$), and whose squares are zero. The product of the forms will then be equal to the invariant in question, multiplied by the product of all the variables. The quartic forms may be represented by the symbol , the quadrics by . Thus the invariant

$$(xyzu)\,(yzvw)\,(xv)\,(uw)$$

will be represented by the figure ; whereas $(xyzu)\,(yzvw)\,(xu)\,(vw)$ is this form  . The former is clearly the product of the two quartic covariants got by cutting it across the dotted lines; while the latter is the product of the quadric covariants  . A *bond* between two forms means a set of variables common to them. Of course we may regard two or more forms as identical, and so form

invariants of a single form; thus is the discriminant of a cubic.... Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again—what you call unravelment."

Besides this passage, the only remains of Prof. Clifford's writings on this subject are contained in a few fragmentary jottings, made at irregular intervals during the last year or two of his life. He often spoke about his progress in the subject; but, knowing how little strength he had to spare, and hoping that he might have written some fuller and more connected account of it, his friends naturally abstained from asking particulars on a variety of points, which would have now been invaluable. The importance which he himself attached to the method, the amount of attention which he gave to it, and the power which it would manifestly give to any one capable of wielding the master's weapon, will perhaps be sufficient apology for offering to mathematicians the very incomplete notes here collected.

The subject of Prof. Clifford's investigations was that of binary quantics, so that the number of variables in each of his sets, viz., the number of x's, y's, &c., is two; say, x_1, x_2; y_1, y_2, &c. A linear form will then, when written in full, be represented thus:

$$a_1x_1+a_2x_2;$$

a quadric form thus:

$$a_{11}x_1y_1+a_{12}x_1y_2+a_{21}x_2y_1+a_{22}x_2y_2;$$

and so on. Between any two of the sets, *e.g.*, between the x's and the y's, we may form the determinant $x_1y_2-x_2y_1$, an expression which will be of frequent occurrence.

To find the product of a form by the determinant of some of its variables. If the form be linear, we have

$$(a_1x_1+a_2x_2)(x_1y_2-x_2y_1) = (x)\,|x, y|\,, \text{ suppose};$$

and effecting the multiplication, while bearing in mind that the variables are to combine as alternate numbers, the product will be found to be

$$= -x_1x_2\,(a_1y_1+a_2y_2),$$

and if we put $x_1x_2 = y_1y_2 = \ldots 1$, the result of multiplying the form by the determinant will be the same as making x identical with $-y$.

If the form be quadric, we have

$$(xy)\,|\,x, y\,| = -(a_{12}-a_{21}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

i.e., the effect of the multiplication will be, just as in the case of the linear form, the same as making x identical with $-y$. Similarly, for

a cubic $(xyz) = a_{111}x_1y_1z_1+\ldots$, we should find

$$(xyz)\,|\,x, y\,| = (a_{211}-a_{121})\,z_1+(a_{212}-a_{122})\,z_2 \ldots\ldots\ldots\ldots(3);$$

and the same is obviously true with forms of any order.

It is further to be observed that the effect of multiplication of a symmetric form by the determinant of any two sets of its variables will be to reduce the expression to zero. Thus, if the quadric be symmetrical, $a_{12}=a_{21}$, and the expression (2) vanishes. If the cubic be symmetrical, $a_{211}=a_{121}$, $a_{212}=a_{122}$, and the product (3) will then vanish. And, generally, if $((x, y, z, \ldots))$ represent the symmetrical form of which $(x, y, z, \ldots)$ is the general form, the same form written as a quadric $((x, y))$, in which the coefficients involve the other variables, will be symmetrical; and consequently

$$((x, y, z, \ldots))\,|\,x, y\,| = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Lastly, it will be convenient to remark that the square of any form of an odd order will vanish; for any such form may be written as a linear form in respect of one of its sets of variables, thus

$$(x, y, \ldots v, w) = (x, y, \ldots v)_1 w_1+(x, y, \ldots v)_2 w_2,$$

in which $(x, y, \ldots v)_1$, $(x, y, \ldots v)_2$ are forms of an even order. Hence, by known properties of alternate numbers,

$$(x, y, \ldots v, w)^2 = (x, y, \ldots v)_1\,(x, y, \ldots v)_2\,(w_1w_2+w_2w_1) = 0\ldots(5).$$

We now come to the forms themselves. In what follows, the main object has been to collect together the fragments which remain of Prof. Clifford's "Graphs"; and this seems to be all that can be done for the present purpose. But, with a view to a better understanding of the very novel method, I have ventured to prefix to each section an algebraical statement of the question.

Linear Forms.

Let $(x) = a_1x_1+a_2x_2$; and, when there are two such forms with different coefficients, let

$$(x)_1 = a_{11}x_1+a_{21}x_2,$$
$$(x)_2 = a_{12}x_1+a_{22}x_2.$$

Then the cases which we have to consider are

$$(x)^2,\ (x)\,(y),\ (x)_1\,(x)_2.$$

Of these, the first $(x)^2 = 0$, in virtue of the general remark made above. The second

$$(x)\,(y) = a_1^2\,x_1y_1+a_1a_2\,(x_1y_2+x_2y_1)+a_2^2\,x_2y_2,$$

in which the coefficients a_1, a_2 may represent either ordinary quantities, or forms of alternate numbers of even orders. If, however, the coefficients represent forms of odd orders, then

$$(x)(y) = -a_1 a_2 (x_1 y_2 + x_2 y_1).$$

Again, $(x)_1 (x)_2 = a_{11} a_{22} - a_{21} a_{12} = D$, suppose,

or it will $= -D$, if the coefficients represent forms of odd orders.

Lastly, as shown in the preceding section,

$$(x) \mid x, y \mid = -(y).$$

In graphical language, (x) will be represented by the symbol $\circ\!-$; and the square of it by $\circ\!-\!\circ$; so that we shall have the relation

$$\circ\!-\!\circ = 0.$$

And if $(x)_1 =$ ①–, and $(x)_2 =$ ②–, then

$$①\!-\!② = D,$$

where D represents the determinant of the two linear forms.

Again, if the determinant $\mid x, y \mid$, be represented by $\begin{matrix} x \\ y \end{matrix}$), or by) where no ambiguity can arise, then we have the relation

$$\circ\!-\!x\ ) = -\!\circ\!-\!y.$$

Quadric Forms.

Algebraically, the form will be represented by (x, y); and the combinations which we have to consider are

$$(x, y) \mid x, y \mid,\ (x, y) \mid y, z \mid,\ (x, y)^2,\ (x, y)(x, z).$$

In virtue of the general remark I made above, in order to evaluate the first of these, we have only to replace y by $-x$; the result will then easily be seen to be

$$(x, y) \mid x, y \mid = a_{21} - a_{12}, = c \text{ suppose.}$$

Again, in virtue of the general theorem II.,

$$(x, y) \mid y, z \mid = -(x, z).$$

Also $$(x, y) = (x)_1 y_1 + (x)_2 y_2,$$

in which $(x)_1$, $(x)_2$, are linear functions of x; hence

$$(x, y)^2 = -2\,(x)_1\,(x)_2 = -2D.$$

Again,
$$\begin{aligned}(x, y)(x, z) &= \{(x)_1 y_1 + (x)_2 y_2\}\,\{(x)_1 z_1 + (x)_2 z_2\} \\ &= -(x)_1 (x)_2 \mid y, z \mid \\ &= -D \mid y, z \mid .\end{aligned}$$

Proceeding further, we might form combinations, such as

$$(x, y)\,(x, z)\,(y, z) = -D|\,y, z\,|\,(y, z)$$
$$= -CD.$$

Or, again, $(x, y)\,(x, z)\,(u, y)\,(u, z) = (x, y)(x, z)\,.\,(u, y)\,(u, z)$
$$= 2D^2.$$

From which it appears that no new forms are to be obtained.

To these may be added, if

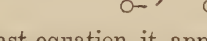
$$(x, y)_1 = a_{111}x_1y_1 + a_{121}x_1y_2 + a_{211}x_2y_1 + a_{221}x_2y_2,$$
$$(x, y)_2 = a_{112}x_1y_1 + a_{122}x_1y_2 + a_{212}x_2y_1 + a_{222}x_2y_2\,;$$

then $(x, y)_1\,(x, y)_2 = -a_{111}a_{222} + a_{121}a_{212} + a_{211}a_{122} - a_{221}a_{112}$
$$= -D_{12}, \text{ suppose.}$$

Graphically, the form itself will be represented by –○– ; –○–), or ○◯, will then represent the invariant $a_{21} - a_{12}$, which vanishes when the form is symmetrical. ○=○ will then represent the discriminant, say

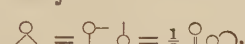
$$○=○ = -2D.$$

Again, –○–○– will be a quadri-covariant. But, multiplying this by), and remembering that the square of the determinant) is equal to 2, it follows that



From the last equation it appears that this invariant vanishes when the variables are made identical.

Proceeding further, we may add



which also vanishes when the form is symmetrical.

Again,

and so on. Hence these and all other ulterior derived forms are only products and powers of the form itself and its discriminant.

Cubic Forms.

Algebraically, the form will be represented by (x, y, x). In this we may change one, two, or three letters; hence we have the following four forms :—

$$(x, y, z),$$
$$(x, y, w),$$
$$(u, v, z),$$
$$(u, v, w).$$

The form being of an odd order, its square will vanish, viz.,

$$(x, y, z)^2 = 0.$$

The product of the form by the determinant of two sets of its variables gives a linear covariant, as mentioned above, viz.,

$$(x, y, z) \,|\, x, y \,| = (a_{211} - a_{121})\, z_1 + (a_{212} - a_{122})\, z_2,$$

which vanishes when the form is symmetrical.

Of binary products we have three forms, viz.,

$$(x, y, z)(x, y, w) = (u, v, z)(u, v, w),$$
$$(x, y, z)(u, v, z) = (x, y, w)(u, v, z),$$
$$(x, y, z)(u, v, w) = (x, y, w)(u, v, z).$$

Of these, we may write the first thus,

$$(x, y, z) = (x, y)_1 z_1 + (x, y)_2 z_2,$$
$$(x, y, u) = (x, y)_1 u_1 + (x, y)_2 u_2.$$

Hence

$$\begin{aligned}(x, y, z)(x, y, u) &= (x, y)_1^2 z_1 u_1 + (x, y)_1 (x, y)_2 (z_1 u_2 + z_2 u_1) + (x, y)_2^2 z_2 u_2 \\ &= -\{2D_{11} z_1 u_1 + D_{12}(z_1 u_2 + z_2 u_1) + 2D_{22} z_2 u_2\} \\ &= -2H.\end{aligned}$$

Again,
$$\begin{aligned}(x, y, z)(x, u, v) &= \{(y, z)_1 x_1 + (y, z)_2 x_2\}\{(u, v)_1 x_1 + (u, v)_2 x_2\} \\ &= (y, z)_1 (u, v)_2 - (y, z)_2 (u, v)_1.\end{aligned}$$

But
$$-2\,(y, z)_1 = (y, z)_1 \,|\, y, z \,|^2 = (a_{121} - a_{112}) \,|\, y, z \,|,$$
$$-2\,(y, z)_2 = (y, z)_2 \,|\, y, z \,|^2 = (a_{221} - a_{212}) \,|\, y, z \,|.$$

Hence

$$\begin{aligned}4\,(x, y, z)(x, u, v) &= (a_{121} - a_{112})(a_{221} - a_{212})\{\,|\, y, z \,|\,|\, u, v \,| - |\, y, z \,|\,|\, u, v \,|\} \\ &= 0.\end{aligned}$$

Of ternary products we have four forms, viz.,

$$\begin{aligned}&\cdot \quad (x, y, w)(u, v, z)(u, v, w),\\ &(x, y, z) \quad \cdot \quad (u, v, z)(u, v, w),\\ &(x, y, z)(x, y, w) \quad \cdot \quad (u, v, w),\\ &(x, y, z)(x, y, w)(u, v, z) \quad \cdot\end{aligned}$$

These are all really the same, as each involves three of the variables twice, and three of them once. They therefore represent the cubi-covariant.

The cubi-covariant is symmetrical, for it is reduced to zero by the factor, *e.g.*, $|\, u, w \,|$. In fact, taking the last form

$$(x, y, z)(x, y, w)(u, v, z) \,|\, u, w \,| = (x, y, z) \,.\, (x, y, w)(w, v, z),$$

which is equal to the product of the cubic itself into a quadri-covariant, which, as was already shown, vanishes.

Of quaternary products we have only one form, viz.,

$$(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w),$$

which involves each of the variables twice, and is consequently an invariant; viz., it is the discriminant.

The above written forms give an immediate proof of the following theorems. The Hessian of the cubi-covariant may be formed by multiplying together two of the four forms; but

$$(x, y, w)\ (u, v, z)\ (u, v, w) \cdot (x, y, z)\ (u, v, z)\ (u, v, w)$$
$$= -\ (x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w) \cdot (u, v, z)\ (u, v, w).$$

That is to say, the Hessian of the cubi-covariant is equal to the discriminant of the cubic multiplied by the Hessian of the cubic. Again, to form the discriminant of the cubi-covariant, we must multiply all the four forms together; but this product

$$= \{(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w)\}^3.$$

That is to say, the discriminant of the cubi-covariant is equal to the cube of the discriminant of the cubic. And again, to form the cubi-covariant of the cubi-covariant, we must multiply together three of the four forms; but, taking the last three, this product

$$= (x, y, z)\ \{(x, y, z)\ (x, y, w)\ (u, v, z)\ (u, v, w)\}^2.$$

That is to say, the cubi-covariant of the cubi-covariant is equal to the product of the cubic itself into the square of the discriminant of the cubic.

We may consider also the case of

$$(y, z, u)\ (z, x, v)\ (x, y, w).$$

But $(y, z, u)\,|\,y, z\,| = (a_{211} - a_{121})\,u_1 + (a_{212} - a_{122})\,u_2 = (u)$, suppose; with similar transformations of the other factors. Hence, operating a second time with $|y, z|$, $|z, x|$, $|x, y|$, on the three factors respectively, we obtain

$$-8\ (y, z, u)\ (z, x, v)\ (x, y, w) = (u)\ (v)\ (w)\,|\,y, z\,|\,|\,z, x\,|\,|\,x, y\,|.$$

But the product of the last three factors $= -2$;* hence

$$4\ (y, z, u)\ (z, x, v)\ (x, y, w) = (u)\ (v)\ (w).$$

And consequently the covariant vanishes when the original cubic is symmetrical. The covariant itself is also obviously symmetrical.

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* See Spottiswoode on "Determinants of Alternate Numbers," Proceedings of the London Mathematical Society, Vol. VII., Nos. 94 and 95.

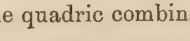
Again, the form

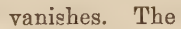
$$4\,(y, z, u)\,(z, x, v)\,(x, y, v) = (u)\,(v)^2 = 0.$$

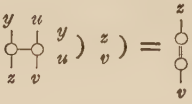
Hence this covariant vanishes.

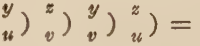
Graphically, the cubic form will be represented by ; and then we shall have

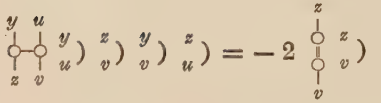
representing a linear covariant which vanishes when the form is symmetrical. The quadric combinations will be represented by



Of these the first vanishes. The second represents the Hessian, or $= -2H$. And since  it follows that the Hessian is symmetrical. As to the third, writing it explicitly, we have



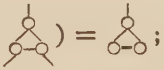
But 

Hence 

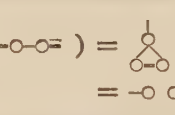
Hence, when the variables are made identical, this vanishes.

Passing to ternary combinations, we have the forms

The first of these is obviously symmetrical; and it is therefore reduced to zero by). But

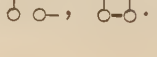


and consequently the second form vanishes. Moreover, since

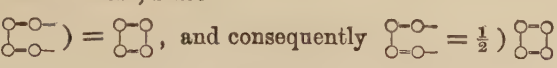


it follows that this form is symmetrical. It is the cubi-covariant Φ.

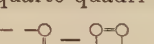
For combinations of four we have only the two forms



As to the first of these, since

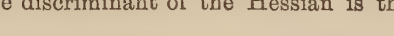


it follows that there is no quarto-quadri-covariant. And as to the second, since

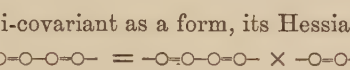


it follows that the discriminant of the Hessian is the discriminant of the cubic.

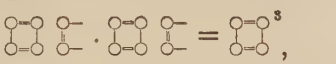
If we take the cubi-covariant as a form, its Hessian will be



But

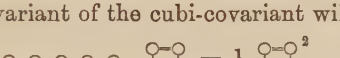


in other words, as shown algebraically, the Hessian of the cubi-covariant is the Hessian of the cubic multiplied by the discriminant of the cubic. Its discriminant will be



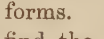
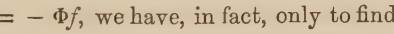
as proved algebraically.

Lastly, the cubi-covariant of the cubi-covariant will be



as before.

The only irreducible form of the third degree being the cubi-covariant, the forms of the fourth degree are its alliances with the cubic; of these the only irreducible one is an invariant, and there are therefore no more irreducible forms.

If we wish to find the Hessian of the compound form $\kappa f + \lambda\Phi$, we shall have to join κ $+\lambda$  to itself by two bonds. And bearing in mind that $f\Phi = -\Phi f$, we have, in fact, only to find the Hessian of Φ. This is found by the theory of quadric forms, applied to Δ, to be $\frac{1}{2}H\Delta$; viz.,  . Therefore the Hessian in question, say $H_{\kappa,\lambda} = \kappa^2 H + \frac{1}{2}\lambda^2 \nabla H = H(\kappa^2 + \frac{1}{2}\lambda^2\nabla) = H\Theta$, suppose. To find $\Phi_{\kappa,\lambda}$, we must join $\kappa f + \lambda\Phi$ to the above by one bond. The result is

$$(\kappa\Phi - \tfrac{1}{2}\lambda\nabla f) = \tfrac{1}{2}\Theta(\Phi\vartheta_\kappa - f\vartheta_\lambda)\Theta.$$

Quartic Forms.

The form itself will be represented by (x, y, z, t). Then

$$(x, y, z, t)^2 = \text{the quadric invariant} = I.$$

Again, $(x, y, z, t)(x, y, z, s) = (x, y, z)'(x, y, z)''|t, s|$;

and consequently this covariant vanishes when the variables are made

identical. Next,

$$(x, y, z, t)\,(x, y, w, s) = (z, t, w, s)_1 = \text{Hessian} = H,$$
$$(x, y, z, t)\,(x, v, w, s) = x_1 x_2\,|\,(y, z, t)_1\,(v, w, s)\,|\,;$$

and consequently this covariant also vanishes when the variables are made identical.

Next, for cubic covariants,

$$(x, y, z, t)\,(x, y, w, s)\,(z, t, w, s) = \text{cubic invariant} = J,$$
$$(x, y, z, t)\,(x, y, w, s)\,(z, v, w, s) = (z, t, w, s)_1\,(z, v, w, s) = (t)_1\,(v)$$
$$= (x, y, z, v)_1\,(x, y, z, t) = (v)_1\,(t).$$

But if $(w) = b_1 v_1 + b_2 v_2$, $(v)_1 = c_1 v_1 + c_2 v_2$,

the condition $(t)_1\,(v) = (v)_1\,(t)$ will give

$$t_1 v_2 - v_1 t_2 = 0, \text{ or } t_1 v_2 + t_2 v_1 = 0\,;$$

and consequently $(t)_1\,(v)$, or $(v)_1\,(t)$, will contain $|\,t, v\,|$ as a factor, and the covariant will consequently vanish when the variables are made identical.

Again,

$$(x, y, z, t)\,(x, y, w, s)\,(u, v, w, s) = (z, t, w, s)_1\,(u, v, w, s) = (z, t)'_1\,(u, v)'$$
$$= (u, v, x, y)_1\,(z, t, x, y) = (u, v)'_1\,(z, t)$$

contains factors $|\,z, u\,|$, $|\,t, v\,|$; or $|\,z, v\,|$, $|\,t, u\,|$; and consequently vanishes when the variables are made identical.

Next, $(x, y, z, t)\,(x, y, w, s)\,(u, v, k, s)$;

which does not admit of any such double form as the two preceding products. This last is the sextic covariant, Φ.

Binary Forms of Alternate Variables. *By the late* Prof. W. K. Clifford, F.R.S.

[*Read June 12th*, 1879.]

Introduction.

1. Alternate numbers are such that $\alpha\beta = -\beta\alpha$, $\alpha^2 = 0$, $\beta^2 = 0$. It is easily shown that linear functions of them possess the same properties; *i.e.*, if $\overline{\alpha} = a_1\alpha_1 + a_2\alpha_2 + \ldots$, $\overline{\beta} = b_1\beta_1 + b_2\beta_2 + \ldots$, where the a, b are scalars and the α, β alternate numbers, then we shall have $\overline{\alpha}\,\overline{\beta} = -\overline{\beta}\,\overline{\alpha}$, and $\overline{\alpha}^2 = 0 = \overline{\beta}^2$. If M, N are homogeneous functions of alternate numbers of degrees m, n respectively, the number of interchanges of

consecutive letters necessary to pass from MN to NM is mn; thus we have

$$MN = (-)^{mn} NM.$$

Or the product of two functions changes sign when the order of the functions is changed *only when their degrees are both odd;* that is to say, forms of odd degree among themselves behave like alternate numbers, forms of even degree in all cases like scalars. It follows that the square of any form of odd degree is zero.

Determinants of alternate numbers.

2. In expanding a determinant of alternate numbers, the order of the *rows* must be followed in multiplication; that is to say, in every term of the expanded determinant the constituent from the first row must be written first, that from the second row second, and so on. Thus, in expanding $\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = (\lambda\mu\nu)$, the terms are of the form $\pm \lambda_r \mu_s \nu_t$, where rst is a permutation of 123 and the signs follow the ordinary rule. An interchange of two *columns* will then alter the sign of the determinant, but an interchange of two rows will leave it unaltered. For the change of sign caused by the interchange is in the latter case counteracted by the change in the order of multiplication. Thus the determinant $(\lambda\mu\nu)$ written above is a symmetrical function of the $\lambda\mu\nu$.

3. Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n-fold space), and in that case it is convenient to regard the product of all the numbers of any set as equal to unity. Hence the products of all but one of the numbers make a new set, the *reciprocal* numbers. The n^{th} power of a determinant of *even* order n is $-|\underline{n}$; the $\overline{n-1}^{\text{th}}$ power is the determinant of the reciprocal numbers. Considering especially determinants of the second order, we have an important theorem of their multiplication, viz., $(\lambda\mu)(\mu\nu) = -(\lambda\nu)$. For

$$\begin{aligned}(\lambda_1\mu_2 - \lambda_2\mu_1)(\mu_1\nu_2 - \mu_2\nu_1) &= \lambda_1\mu_2\mu_1\nu_2 + \lambda_2\mu_1\mu_2\nu_1 \\ &= (\lambda_2\nu_1 - \lambda_1\nu_2)\,\mu_1\mu_2 \\ &= -(\lambda_1\nu_2 - \lambda_2\nu_1).\end{aligned}$$

Hence $(\lambda\mu)(\mu\nu)(\nu\rho)\ldots(\sigma\tau)$ [n factors] $= (-)^{n-1}(\lambda\tau)$.

An analogous theorem holds for determinants of the n^{th} order; viz., if we denote a determinant with r rows of λ and s rows of μ by $(\lambda^r\mu^s)$, where $r+s=n$; then

$$(\lambda^r\mu^s)(\mu^r\nu^s)\ldots(\sigma^r\tau^s)\ [k \text{ factors}] = (-)^{s(n+r)k}(\lambda^r\tau^s) = (-)^{sk}(\lambda^r\tau^s),$$

since $n+r = 2r+s \equiv (\text{mod. } 2)$, and so $s(n+r) \equiv s^2 \equiv s \ (\text{mod. } 2)$.

Multipartite Forms.

4. A homogeneous function linear in each of n sets of k alternate numbers is called a k-ary multipartite form of the n^{th} order, or shortly, a k-ary form of the n^{th} order. We may consider also forms of any order lower than k in any of the sets; but for the present we restrict ourselves to the case in which the forms are linear. Consider now k forms, each linear in regard to the k alternate numbers $\lambda_1, \lambda_2, \ldots \lambda_k$; viz.,

$$\begin{aligned} F_a &= a_1\lambda_1 + a_2\lambda_2 + \ldots + a_k\lambda_k \\ &\vdots \\ F_k &= k_1\lambda_1 + k_2\lambda_2 + \ldots + k_k\lambda_k, \end{aligned}$$

where the coefficients a, b, ... k are themselves k-ary multipartite forms of alternate numbers.

The product $F_aF_b \ldots F_k$ is an invariant; that is to say, if for the λ we substitute k linear functions of them, say the μ, then the functions F will be transformed into functions of the μ; and if we form the same function of the new coefficients that ΠF is of the old coefficients, one will be equal to the other multiplied by the determinant of transformation.

For the product is $I.\lambda_1\lambda_2 \ldots \lambda_k$, whether we regard the λ as linear functions of the μ or not; but in the latter case

$$\lambda_1\lambda_2 \ldots \lambda_k = D.\mu_1\mu_2 \ldots \mu_k,$$

where D is the determinant of transformation.

But it is also to a constant factor *près* the only function possessing this property. For let I be such a function, and calculate it for the linear forms $\lambda_1, \lambda_2, \ldots \lambda_k$. As all the coefficients are here either unity or zero, I must be represented by a constant, say I_0. Now, expressing the λ in terms of the μ, we have, by hypothesis, $I_\mu = DI_0 = \lambda_1 \ldots \lambda_k . I_0$.

It is to be remarked that the coefficients of transformation may themselves be forms involving alternate numbers to any *even* order. Otherwise the μ would not be alternate numbers, which is implied.

Moreover, we may regard the λ as no longer linear forms, but forms of any odd order; the new invariant will then be equal to the old one multiplied by a *commutant* of transformation. This leads to a useful theorem in the comparison of invariants; *e.g.*,

$$c|456\mathfrak{s}.a|\mathfrak{s}2.b|\mathfrak{s}'3.c|456\mathfrak{s}' = (c|456\mathfrak{s}|^2.a|\mathfrak{s}'2.b|\mathfrak{s}'3.$$

The proposition may be further extended by considering forms which involve the λ to an order higher than the first, but less than k; *i.e.*, linear functions of their products r together. Let $F_a, F_b, \ldots F_k$ be forms such that the sum of their orders in the λ is equal to k; then their product is an invariant (and the only one) in regard to linear transformations of the λ.

If the sum of the orders is less than k, $= h$ suppose, the product is a covariant; viz., it is a linear function of the products of the λ, h together, which, whether derived from the original forms before or after the λ are replaced by linear functions of the μ, has the same value. In this case the r-products of the λ are replaced by linear functions of the r-products of the μ, the coefficients being determinants of the r^{th} order formed with the coefficients of the μ.

Now suppose any number of forms $F_a \ldots F_n$ to involve any number of sets of alternate nnmbers $\lambda\mu \ldots \tau$, yet so that the sum of the orders in any one set is not greater than the number of alternates in that set; then the product of the forms is an invariant or covariant in regard to each of the sets taken separately, in the sense explained above; and it is the only function which possesses this property.

Expression of Unsymmetrical forms in terms of Symmetrical forms and Determinants of the Variables.

5. The binary form in two sets of variables,

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = a12,$$

will be called *symmetrical* when $a_{12} = a_{21}$. In that case the interchange of the variables λ, μ only alters the sign of the form; we have $a12 = -a21$. We have, in general,

$$a12 + a21 = (a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1) = s\,(\lambda\mu).$$

The factor $a_{12} - a_{21}$ is an invariant when both sets of variables are transformed by the same substitutions; in fact, we have

$$(\lambda_1\mu_2 - \lambda_2\mu_1)\, a12 = a_{21} - a_{12} = -s,$$

which exhibits it as a product of the form by the universal covariant $(\lambda\mu)$ or (12). We may make a symmetrical form from $a12$ by adding $-a21$ to it; half this sum shall be called the mean value of $a12$ and denoted by $\overline{a12}$. Thus, we have

$$\overline{a12} = \tfrac{1}{2}(a12 - a21) = a_{11}\lambda_1\mu_1 + \tfrac{1}{2}(a_{12} + a_{21})(\lambda_1\mu_2 + \lambda_2\mu_1) + a_{22}\lambda_2\mu_2;$$

but also $\tfrac{1}{2}(a12 + a21) = \tfrac{1}{2}(a_{12} - a_{21})(\lambda_1\mu_2 - \lambda_2\mu_1)$,

therefore $a12 = \overline{a12} + \tfrac{1}{2}s\,.\,(12)$,

where $-s = (12)\, a12$.

It is easy to apply this to forms involving more sets of variables, if we remember that in these results the coefficients may themselves be such forms. We have, for example,

$$a123 = \overline{a123} - \tfrac{1}{3}[(23)\, a\,.\,23 + (13)\, a\,.\,(13) + (12)\, a\,.\,(12)],$$

and so, generally,

$$a12\ldots k = \overline{a12\ldots k} - \frac{\Sigma\,(12)\,a\,.\,(12)}{n} + \frac{\Sigma\,(12)(34)\,a\,.\,(12)\,(34)}{\frac{1}{2}n\,(n-1)} - \ldots,$$

the coefficients being the reciprocals of the binomial coefficients.

Theory of Quadratic Forms.

6. In connection with the quadratic form $a12$, we have already considered the invariant

$$a12 + a21 = s_a\,(12), \text{ where } (12)\,a12 = -\,s_a.$$

We have thus the formula

$$a21 = -\,a12 + s_a\,.\,(12) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

To this if we add

$$(13)\,a12 = -\,a32, \quad (23)\,a12 = -\,a13 \quad \ldots\ldots\ldots\ldots (2),$$

we shall have exhausted all the invariants and covariants of the first order in the coefficients.

The *discriminant*, which is the only invariant of the second order, is given by the square of the quadratic form. We may write

$$a12\,.\,a12 = -\,2D_{aa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where $$D_{aa} = a_{11}a_{22} - a_{12}a_{21}.$$

We have
$$\begin{aligned} a21\,.\,a21 &= [-a12 + s_a\,.\,(12)]\,[-a12 + s_a\,.\,(12)] \\ &= a12\,.\,a12 - 2s_a\,(12)\,a12 + s_a^2\,.\,(12)^2 \\ &= a12\,.\,a12 - 2s_a^2 + 2s_a^2 = a12\,.\,a12, \end{aligned}$$

as it obviously should; and this may be regarded as a proof of the formula (2).

Moreover,
$$\begin{aligned} a12\,.\,a21 &= a12\,[-a12 + s_a\,(12)] = -\,a12\,.\,a12 - s_a^2 \\ &= 2D_{aa} - s_a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4); \end{aligned}$$

and again, since
$$\overline{a12} = a12 - \tfrac{1}{2}s\,.\,(12),$$

$$\begin{aligned} \overline{a12}\,.\,\overline{a12} &= a12\,.\,a12 + s_a^2 + \tfrac{1}{4}s_a^2\,(12) \\ &= a12\,.\,a12 + \tfrac{1}{2}s_a^2; \end{aligned}$$

therefore
$$\overline{D_{aa}} = D^{aa} - \tfrac{1}{4}s_a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $\overline{D_{aa}}$ is the corresponding invariant of the symmetrical function $\overline{a}$.

Since the product of two *even* forms is independent of their order, we have
$$a12\,.\,a13 = a13\,.\,a12.$$

But $$a12\,.\,a13+a13\,.\,a12 = -\,(23)\,a12\,.\,a13\,.\,(23)$$
$$= a13\,.\,13\,(23) = -\,2D_{aa}\,(23),$$

therefore $$a12\,.\,a13 = -\,D_{aa}\,(23) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Hence $$a12\,.\,a31 = a12\,[-a13+s_a\,(13)] = D_{aa}\,.\,(23) - s_a\,.\,a32 \quad \ldots(7),$$

$$a21\,.\,a31 = [-a12+s_a\,(12)]\,[-a13+s_a\,(13)]$$
$$= -\,D_{aa}\,(23)+s_a\,(a32+a23)-s_a^2\,(23)$$
$$= -\,D_{aa}\,(23) = a12\,.\,a13 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$a21\,.\,a13 = a21\,[-a31+s_a\,(13)] = D_{aa}\,(23) - s_a\,.\,a23 \ldots\ldots (9).$$

In the case of a symmetrical function $s = 0$, and the formulæ reduce themselves to the following:

$$a12\,a12 = -\,2D_{aa},\quad a21 = -\,a12,\quad a12\,.\,a13 = -\,D_{aa}\,.\,(23).$$

The following formulæ are also worth noticing:

$$a12\,.\,a34+a13\,.\,a24 = [D_{aa}\,.\,(14) - s_a\,.\,a14]\,(23) \quad \ldots\ldots (10),$$

$$a12\,.\,a34-a21\,.\,a43 = s_a\,[a23\,.\,(14)-a14\,.\,(23)]$$
$$= s_a\,[a41\,.\,(23)-a32\,.\,(14)] \quad \ldots\ldots (11);$$

these enable us to make a single or bifid substitution in a product of the form by itself with two different sets of variables.

Passing now to two different forms a, b, we have in the first place the covariant

$$\vartheta_{ab}23 = a12\,.\,b13 = \begin{pmatrix} a_{11}b_{21}-a_{21}b_{11}, & a_{11}b_{22}-a_{21}b_{12} \\ a_{12}b_{21}-a_{22}b_{11}, & a_{12}b_{22}-a_{22}b_{12} \end{pmatrix} (\mu_1\mu_2)(\nu_1\nu_2) \ldots (12),$$

and next the invariant

$$D_{ab} = a_{11}b_{22}-a_{12}b_{21}-a_{21}b_{12}+a_{22}b_{11} = -\,a12\,.\,b12 \quad \ldots\ldots (13).$$

We may now make the following analysis of the one-place and two-place products of a and b:

$$\left.\begin{aligned} &a12\,.\,b13 = \vartheta 23, \\ &a12\,.\,b31 = a12\,[-b13+s_b\,(13)] = -\,\vartheta 23 - s_b\,.\,a32, \\ &a21\,.\,b13 = [-a12+s_a\,(12)]\,b13 = -\,\vartheta 23 - s_a\,.\,b23, \\ &a21\,.\,b31 = [-a12+s_a\,(12)]\,[-b13+s_b\,(13)] \\ &\qquad = \vartheta 23 + s_b\,.\,a32 + s_a\,.\,b23 - s_a s_b\,.\,23 \end{aligned}\right\} \ldots (14);$$

$$\left.\begin{aligned} &a12\,.\,b12 = -\,D_{ab}, \\ &a12\,.\,b21 = a12\,[-b12+s_b\,.\,(12)] = D_{ab} - s_a s_b \end{aligned}\right\} \quad \ldots\ldots\ldots (15).$$

Let us now apply to ϑ all those formulæ which we have proved for

a single form. Thus, from (1), we have

$$\mathfrak{Y}23+\mathfrak{Y}32=-(23)\mathfrak{Y}23(23)=-(23)\,a12\,.\,b13\,.(23)=a13\,.\,b13\,.(23)$$
$$=D_{ab}(23)\text{ or }s_{\mathfrak{Y}}=D_{ab}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

Next, from (3),

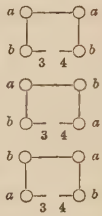
$$-2D_{\mathfrak{YY}}=\mathfrak{Y}23\,.\,\mathfrak{Y}23=a12\,.\,b13\,.\,a42\,.\,b43=D_{aa}D_{bb}(14)^2$$
$$=-2D_{aa}D_{bb}\text{ or }D_{\mathfrak{YY}}=D_{aa}D_{bb}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Hence
$$\left.\begin{aligned}\mathfrak{Y}23\,.\,\mathfrak{Y}32&=2D_{aa}D_{bb}-D_{ab}^2,\\ \overline{\mathfrak{Y}23}&=\mathfrak{Y}23-\tfrac{1}{2}D_{ab}(23),\\ \text{and}\qquad \overline{D_{\mathfrak{YY}}}&=D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2,\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

which last agrees with the known formula.

We have next the one-place products of $\mathfrak{Y}$ by itself; namely,

$$\left.\begin{aligned}\mathfrak{Y}23\,.\,\mathfrak{Y}24&=-D_{aa}D_{bb}(34)=\mathfrak{Y}32\,.\,\mathfrak{Y}42=a12\,.\,b13\,.\,a52\,.\,b54\\ &\qquad\qquad a13\,.\,b12\,.\,a54\,.\,b52\\ \mathfrak{Y}23\,.\,\mathfrak{Y}42&=+D_{aa}D_{bb}(34)-D_{ab}\,.\,\mathfrak{Y}43=a12\,.\,b13\,.\,a54\,.\,b52\\ \mathfrak{Y}32\,.\,\mathfrak{Y}24&=+D_{aa}D_{bb}(34)-D_{ab}\,.\,\mathfrak{Y}34=a13\,.\,b12\,.\,a52\,.\,b54\end{aligned}\right\}\quad(19);$$



but it is important also to calculate these for the mean value $\overline{\mathfrak{Y}}$; namely,

we have $\overline{\mathfrak{Y}23}\,.\,\overline{\mathfrak{Y}24}=[\mathfrak{Y}23-\tfrac{1}{2}D_{ab}(23)][\mathfrak{Y}24-\tfrac{1}{2}D_{ab}(24)]$
$$=-D_{a,}D_{bb}(34)+\tfrac{1}{2}D_{ab}(\mathfrak{Y}43+\mathfrak{Y}34)-\tfrac{1}{4}D_{ab}^2(34)$$
$$=-(D_{aa}D_{bb}-\tfrac{1}{4}D_{ab}^2)(34)=-\overline{D_{\mathfrak{YY}}}\,.(34),$$

as it ought to be, confirming equation (18).

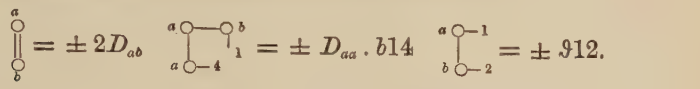
Lastly, from (10) and (11), we get

$$\mathfrak{Y}12\,.\,\mathfrak{Y}34+\mathfrak{Y}13\,.\,\mathfrak{Y}24=[D_{aa}\,.\,D_{bb}\,.(14)-D_{ab}\,.\,\mathfrak{Y}14](23)$$
$$\mathfrak{Y}12\,.\,\mathfrak{Y}34-\mathfrak{Y}21\,.\,\mathfrak{Y}43=D_{ab}[\mathfrak{Y}23\,.(14)-\mathfrak{Y}14(23)]$$
$$=D_{ab}[\mathfrak{Y}41\,.(23)-\mathfrak{Y}32\,.(14)]\quad\ldots\ldots\ldots\ldots(20).$$

There are four one-place products of a, a, b; namely,

$$\left.\begin{aligned}a12\,.\,a32\,.\,b34&=D_{aa}\,.\,b14,\qquad a12\,.\,a32\,.\,b43=D_{aa}\,.\,b41,\\ a12\,.\,a23\,.\,b34&=-D_{aa}\,.\,b14+s_a\,.\,\mathfrak{Y}14+s_a^2\,.\,b14=(s_a^2-D_{aa})\,b14+s_a\,.\,\mathfrak{Y}14,\\ a12\,.\,a23\,.\,b43&=-D_{aa}\,.\,b41-s_a\,.\,\mathfrak{Y}14-s_as_b\,.\,a41-s_a^2b14+s_as_b(14)\\ &=(s_a^2-D_{aa})\,.\,b41-s_a\,.\,\mathfrak{Y}14-s_as_b\,.\,a41,\end{aligned}\right\}\quad\ldots(21).$$

When the forms are symmetrical, all these reduce to



$$= \pm 2D_{ab} \qquad = \pm D_{aa} \,.\, b14 \qquad = \pm \vartheta 12.$$

Three forms a, b, c give rise to the new invariant

$$\begin{aligned} R_{abc} = a12 \,.\, b23 \,.\, c31 = &+ a_{11}(b_{21}c_{22} - b_{22}c_{12}) \\ &- a_{12}(b_{11}c_{22} - b_{12}c_{12}) \\ &- a_{21}(b_{21}c_{21} - b_{22}c_{11}) \\ &+ a_{22}(b_{11}c_{21} - b_{12}c_{11}) \quad \ldots\ldots\ldots(22), \end{aligned}$$

and the remaining closed products are of the type

$$a21 \,.\, b23 \,.\, c31 = [a12 + s_a(12)]\, b23 \,.\, c31 = -R_{abc} + s_a D_{bc} - s_a s_b s_c.$$

R calculated for the symmetrical functions $\overline{a}$, $\overline{b}$, $\overline{c}$ is

$$\begin{aligned} \overline{R_{abc}} &= [a12 - \tfrac{1}{2}s_a(12)][b23 - \tfrac{1}{2}s_b(23)][c31 - \tfrac{1}{2}s_c(31)] \\ &= R_{abc} + \tfrac{1}{2}(s_a D_{bc} + s_b D_{ca} + s_c D_{ab}) - \tfrac{1}{2}s_a s_b s_c \quad \ldots\ldots\ldots\ldots(23). \end{aligned}$$

It may be mentioned in this place that the determinant made with the coefficients of four forms a, b, c, d has the value

$$\text{Det.}\,(a, b, c, d) = s_a R_{bcd} - s_b R_{cda} + s_c R_{dab} - s_d R_{abc} = s_a \overline{R_{bcd}} - \&c.,$$

and that no additional invariants are introduced by a greater number of forms.

We now consider the open one-place products of 3 forms. There are evidently four such products having b in the middle; and they are in the first place connected by the formulæ

$$a12 \,.\, b23 \,.\, c34 + a42 \,.\, b23 \,.\, c31 = -R_{abc}(14),$$

$$a12 \,.\, b23 \,.\, c34 + a12 \,.\, b32 \,.\, c34 = -s_b a13 c34 = s_b \vartheta_{ac} 14 + s_a s_b \,.\, c14,$$

$$a12 \,.\, b23 \,.\, c34 - a34 \,.\, b23 \,.\, c12 = -c32\, a34 \,.\, b21 + c23 \,.\, a21 \,.\, b43.$$

The following presents were made to the Society's Library during the Recess:—

"Leçons sur les Coordonnées Curvilignes et leurs diverses applications;" Paris, 1859.

"Leçons sur les fonctions inverses des Transcendantes et les surfaces Isothermes;" Paris, 1857.

"Leçons sur la Théorie Analytique de la Chaleur;" Paris, 1861.

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Carte-de-Visite portraits of Messrs. C. W. Merrifield, F.R.S., and S. Roberts, F.R.S.

APPENDIX.

FURTHER Papers by Prof. Cayley "On the Double ϑ Functions" (pp. 74—82), and "On the Triple ϑ Functions"* (pp. 134—139), are printed in Crelle's "Journal," 87 Band ("Proceedings," Vol. ix., p. 275).

In the discussion on the "Construction of Magic Squares" (p. 75), reference was made to articles in the "Quarterly Journal of Mathematics," and to other well-known sources. A series of papers on the same subject appeared in the "Queen" for June 21; July 5, 19; August 23, and September 6, 1879.

The fractions alluded to on p. 123, were of the following nature :—

Let $a, b, c, d, e \ldots$ be any numbers excluding unity, and write down the fractions thus— a, b, c, d, e,

$$\frac{a}{1}, \quad \frac{ab+1}{a+b}, \quad \frac{abc+a+b+c}{ab+bc+ca+1} \ldots\ldots$$

The law of formation of the fractions is obvious, and the n^{th} is

(i.) $\dfrac{1+p_2+\ldots+p_n}{p_1+p_3+\ldots+p_{n-1}}$ (n even), or (ii.) its reciprocal (n odd),

where $p_1, p_2, \ldots p_n$ are given by the equation

$$f(x) \equiv x^n - p_1 x^{n-1} + \ldots + (-1)^n p_n = 0,$$

whose roots are the n quantities $a, b, c \ldots$

The results (i.) and (ii.) may also be put into the form $\dfrac{f(1)+f(-1)}{f(-1)-f(1)}$, or its reciprocal.

The other results given were principally those connecting two and three consecutive fractions. Also from the forms $\left(\dfrac{p_n}{q_n}\right.$ denoting the n^{th} fraction$\left.\right)$

$$\left[\frac{p_{n+2r}}{q_{n+2r}} - \frac{p_{n-1}}{q_{n-1}}\right] \div \left[1 - \frac{p_{n-1}}{q_{n-1}} \cdot \frac{p_{n+2r}}{q_{n+2r}}\right] = \frac{1+p_2+p_4+\ldots}{p_1+p_3+\ldots}$$

$$= \left[1 - \frac{p_{n-1}}{q_{n-1}} \cdot \frac{p_{n+2r-1}}{q_{n+2r-1}}\right] \div \left[\frac{p_{n+2r-1}}{q_{n+2r-1}} - \frac{p_{n-1}}{q_{n-1}}\right]$$

* There is also a paper on these by Prof. Cayley in the "Atti della R. Accademia dei Lincei,"—"Transunti," Vol. iii., Fasc. vi. (Maggio), 1879, pp. 190—198.

it was proposed to call the fractions "tangential" fractions from the analogy of the above forms to that for

$$\tan(\alpha+\beta+\gamma+\ldots).$$

Mr. Dickson has furnished the following Abstract of his communication on "the Numerical Calculation of a Class of Determinants, and on Continued Fractions" (p. 130).

1. If

$$\begin{matrix} a_1, & a_2, & a_3 & \ldots \\ b_1, & b_2, & b_3 & \ldots \end{matrix}$$

be two given sets of quantities, from which are deduced new sets

$$\begin{matrix} c_1, & c_2, & c_3 & \ldots \\ d_1, & d_2, & d_3 & \ldots \\ e_1, & e_2, & e_3 & \ldots \\ \ldots & \ldots & \ldots & \end{matrix}$$

by means of auxiliary sets of quantities,

$$\begin{matrix} \alpha_0, & \alpha_2, & \alpha_3, & \alpha_4 & \ldots \\ \beta_0, & \beta_2, & \beta_3, & \beta_4 & \ldots \\ \gamma_0, & \gamma_2, & \gamma_3, & \gamma_4 & \ldots \\ \ldots & \ldots & \ldots & & \end{matrix}$$

where

$$\alpha_0 = \begin{vmatrix} a_1, & 1 \\ b_1, & 1 \end{vmatrix} = a_1 - b_1, \text{ and for } n = 2 \text{ or upwards, } \alpha_n = \begin{vmatrix} a_1, & a_n \\ b_1, & b_n \end{vmatrix},$$

$$\beta_0 = \begin{vmatrix} b_1, & 1 \\ c_1, & 1 \end{vmatrix} = b_1 - c_1, \text{ and for } n = 2 \text{ or upwards, } \beta_n = \begin{vmatrix} b_1, & b_n \\ c_1, & c_n \end{vmatrix},$$

and so on for γ, δ, ...; such that

$$c_1 = \frac{\alpha_2}{\alpha_0}, \quad c_2 = \frac{\alpha_3}{\alpha_0} \ldots\ldots c_n = \frac{\alpha_{n-1}}{\alpha_0},$$

$$d_1 = \frac{\beta_2}{\beta_0}, \quad d_2 = \frac{\beta_3}{\beta_0} \ldots\ldots d_n = \frac{\beta_{n-1}}{\beta_0},$$

and so on for e, f, g ...; then, writing, for example,

$$\Delta_5 = \begin{vmatrix} . & . & b_1, & b_2, & b_3 \\ . & a_1, & a_2, & a_3, & a_4 \\ . & b_1, & b_2, & b_3, & b_4 \\ a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \end{vmatrix}, \quad \Delta_6 = \begin{vmatrix} . & . & a_1, & a_2, & a_3, & a_4 \\ . & . & b_1, & b_2, & b_3, & b_4 \\ . & a_1, & a_2, & a_3, & a_4, & a_5 \\ . & b_1, & b_2, & b_3, & b_4, & b_5 \\ a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & a_2, & b_3, & b_4, & b_5, & b_6 \end{vmatrix},$$

and putting $\Delta_{-1} = -\frac{1}{a_1}$, it can be proved that

$$\frac{\Delta_{-1}\Delta_2}{\Delta_0\Delta_1} = -\frac{a_0 c_1}{a_1 b_1} = -\frac{a_2}{a_1 b_1},$$

$$\frac{\Delta_0\Delta_3}{\Delta_1\Delta_2} = -\frac{\beta_0 d_1}{b_1 c_1} = -\frac{\beta_2}{b_1 c_1},$$

$$\frac{\Delta_1\Delta_4}{\Delta_2\Delta_3} = -\frac{\gamma_0 e_1}{c_1 d_1} = -\frac{\gamma_2}{c_1 d_1}, \text{ \&c.;}$$

whence, putting $\Delta_0 = 1$, $\Delta_1 = b_1$, $\Delta_2 = a_2$, we can calculate successively the values of Δ_3, Δ_4 ... To do this with economy of labour, write down the following series,

a_1,	a_2,	a_3,	...
b_1,	b_2,	b_3,	...
a_2,	a_3,	a_4,	...
$-a_0\beta_2$,	$-a_0\beta_3$,	$-a_0\beta_4$,	...
$\frac{a_0^2\beta_0\gamma_2}{b_1}$,	$\frac{a_0^2\beta_0\gamma_3}{b_1}$,	...	...
$\frac{a_0^3\beta_0^2\gamma_0\delta_2}{b_1 a_2}$,	$\frac{a_0^3\beta_0^2\gamma_0\delta_3}{b_1 a_2}$,	...	...
$\frac{a_0^5\beta_0^3\gamma_0^2\delta_0\epsilon_2}{b_1 . b_1 a_2 . a_0\beta_2}$,	$\frac{a_0^5\beta_0^3\gamma_0^2\delta_0\epsilon_3}{b_1 . b_1 a_2 . a_0\beta_2}$,	...	...
$-\frac{a_0^8\beta_0^5\gamma_0^3\delta_0^2\epsilon_0\zeta_2}{(b_1 a_2)(b_1 . b_1 a_2 . a_0\beta_2)\frac{a_0^2\beta_0\gamma_2}{b_1}}$,	... ...	...	...

where each series is calculated from the two immediately preceding it, in the same way as the a-series is calculated from the a-series and the b-series; with the difference that the result, before being tabulated, is divided by the first term of the series next before the two from which it is calculated.

The first column thus written down, commencing with b, gives the successive values of $\Delta_1 = b_1$, $\Delta_2 = a_2$, $\Delta_3 = -a_0\beta_2$, $\Delta_4 =$ &c., where Δ_n is $+$ except when $n \equiv 3 \pmod 4$.

The saving of labour by this method may be easily shown thus. If $u_n =$ No. of terms in Δ_n, then

$$u_{2m} = \lfloor\underline{m} \cdot \lfloor\underline{m+1},$$

$$u_{2m-1} = \lfloor\underline{m} \cdot \lfloor\underline{m},$$

so that there are 36 terms in Δ_5, 144 in Δ_6, 576 in Δ_7, 2880 in Δ_8, and

so on. Δ_7 is shown above, and Δ_8 may be calculated as easily as Δ_7 has been.

2. It follows from the above that

$$\frac{b_1}{a_1}\,\frac{a_1+a_2x+a_3x^2+\dots}{b_1+b_2x+b_3x^2+\dots} = 1+\cfrac{\cfrac{\Delta_{-1}\Delta_2}{\Delta_0\Delta_1}x}{1+}\;\cfrac{\cfrac{\Delta_0\Delta_3}{\Delta_1\Delta_2}x}{1+}\;\cfrac{\cfrac{\Delta_1\Delta_4}{\Delta_2\Delta_3}x}{1+}\;\cfrac{\cfrac{\Delta_2}{\Delta_3}x}{F_4},$$

where $$F_4 = \frac{\Delta_4+\Delta_{4,1}x+\Delta_{4,2}x^2+\dots}{\Delta_5+\Delta_{5,1}x+\Delta_{5,2}x^2+\dots},$$

$\Delta_{-1} = -\dfrac{1}{a_1}$, $\Delta_0 = 1$; and Δ_4, $\Delta_{4,1}$, ... are the successive terms in the line which commences with Δ_4, viz., $\dfrac{a_0^2\beta_0\gamma_2}{b_1}$; and similarly for Δ_5, $\Delta_{5,1}$, ...; and so on, for further expansions.

Dr. Freeth's Nephroid (p. 130) is generated in the following manner:—

Draw any chord AB of the circle $ABCDE$, centre O, and produce OB to P, making $BP = BA$; the locus of P is the curve required.

Take $OP = r$, $\angle POA = \theta$, a the radius of the circle, then

$$r = OP = OB+AB = a\left(1+2\sin\frac{\theta}{2}\right) \dots\dots\dots\dots\dots\dots\dots\dots\text{(i.)}$$

Bisect OA in F, and erect a perpendicular FP' to meet the outermost branch of the Nephroid in P'; join O and A to P', then OAP' is a triangle having each of its base angles equal three times the vertical angle at P'.

For, let OP' meet the circle in D, then

$$\angle OAD = \angle ODA = 2\angle DAP',$$

and therefore $$\angle OAP' = 3\angle OP'A.$$

It is now easy to inscribe the regular figures of seven, twenty-one, thirty-five, &c. sides, in a circle.

Dr. Freeth refers to the "Philosophical Magazine" for February and for April, 1864, (also cited by "Todhunter, Euclid, Edition 1861,") for an approximate construction of the regular heptagon, and for remarks by Röber, a professor of architecture at Dresden, on the construction of the auxiliary isosceles triangle.

We give here also a brief sketch of the other curves discussed by the same writer (p. 167).

Let the side BC of an equilateral triangle ABC (base AB) be produced, and any point (K) on this line be joined to A; then produce AK to P, making KP equal KB. The locus of P is the curve now to be discussed.

Take $AB = a$, $\angle BAK = \theta$, and the curve is given by the equation

$$r = AP = a \frac{\cos\left(\frac{\theta}{2} - \frac{\pi}{6}\right)}{\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)}.$$

Bisect AB in D, and let DC produced meet the curve in P (say), then APB is a triangle, having each base angle five times the angle at P.

For $$\angle PAC = \angle PBK = \angle BPK,$$

and $$\angle ACB = \angle AKC + \angle CAK = \angle 3CAK;$$

therefore $$\angle CAP = 20^\circ \text{ and } \angle BAP = 80^\circ.$$

The application to the construction of the regular Nonagon is obvious.

This curve, as Dr. Freeth remarks, like the Conchoid and the Trisectrix, can be employed for the trisection of any angle.

The "Supertrisectrix" is thus generated from the Trisectrix:

Let AEC be a trisectrix, of which AFB, centre O, is the generating circle; join O to E, and produce OE to P, so that $EP = AE$, then the locus of P is the curve sought.

Bisecting AO in D, and erecting DP perpendicular to AO to meet the curve in P; it is shown as before that AOP is a triangle, having each base angle five times the vertical angle P. Hence the means of constructing a regular undecagon.

Dr. Freeth's papers were illustrated by numerous large figures.

The above short sketch will sufficiently show the gist and method of the constructions.

The "Bulletin des Sciences Math. et Astron." (Tome iii., Avril, 1879) contains two articles by M. G. Darboux; viz., "Sur un nouvel appareil à ligne droite de M. Hart," pp. 144—151 ("Proceedings," Vol. viii., p. 288), and "Recherches sur un Système Articulé," pp. 151—192 ("Proceedings," Vol. ix., p. 133).

Mr. Kempe has furnished the Secretaries with the following communication:—

"Proofs of the theorem referred to by Professor Cayley* at the Meeting of the Society on June 13th, 1878 (see 'Proceedings,' Vol. ix., p. 148), have been furnished by Mr. A. B. Kempe to the 'American Journal of Mathematics,' Vol. ii., Pt. 3, and 'Nature,' Vol. xxi., p. 399. The latter proof, of which the following is an abstract, is the simpler, and is free from some errors which appeared in the former.

* A paper by Prof. Cayley, "On the Colouring of Maps," is printed in the April (1879) number of the "Proceedings of the Royal Geographical Society."

The theorem is this :—The districts into which any map is divided by its boundaries can be coloured with four colours, so that no two adjacent districts (*i.e.*, districts which have one or more common boundaries, as distinguished from those which are remote from each other, or touch only at detached points) may be coloured the same colour.

(1) By a simple proof it is shown that every map contains at least one pair of adjacent districts which have only one common boundary (β).

(2) If three lines diverge from a point, and one be rubbed out, the remaining two become one line. Hence, if one boundary be rubbed out, it may cause a reduction of the number (B) of boundaries in a map by three, but it cannot cause a greater, and may cause a less.

(3) The obliteration of (β) causes a reduction of the number (D) of districts in a map by one, and the map still remains a map, and has, therefore, a pair of districts having only one common boundary. Obliterate this, and so on. We finally get a single district and no boundary, and each reduction of D by one cannot cause a reduction of B by more than three; thus $3D$ must be greater than B, consequently $6D$ greater than $2B$; but $2B$ is the number arrived at by counting the number of boundaries to each district, and adding them all together; hence it is impossible that every district can have as many as six boundaries, *i.e.*, every map contains at least one district with less than six boundaries.

(4) We can therefore reduce a map to a single district by successive operations of throwing two districts into one, by rubbing out the boundary or boundaries between two districts, of which one has less than six boundaries. Conversely, we can develope a map, starting from a single district and adding boundaries, at each stage dividing a district into two, one of which has less than six boundaries. Suppose, at any stage of its development by this process, a map can be coloured with four colours [red, blue, green, and yellow]. Let these colours be indicated by coloured wafers placed on the districts. Proceed to the next operation; this divides a wafered district into two districts. Shift its wafer on to the district of these two, which is not the one which has less than six boundaries; if both have less than six boundaries, shift the wafer on to either. If the district (W), which is left without a wafer, is only touched by three colours, it can be coloured the fourth, and a wafer may be put on it representing that colour. But if it is touched by all four colours, we must take another step. This can only be necessary if W has four or five adjacent districts. These may either all surround or all be surrounded by, or some surround and some be surrounded by it. Take first the case in which four districts are adjacent, all surrounding or surrounded by W. Let a, b, c, d be the districts, taking them in the order in which they stand. Let a be red, b blue, c green, and d yellow. If, starting from a, we can get to c, going only through red and green districts, and not passing through any corner points, we cannot, starting from b, get to d, going similarly only through blue and yellow districts, for otherwise two tracts which pass through different districts would cross. Thus, b forms one of group (G) of blue and yellow districts, which are cut off from the rest of the map by encircling red and green ones. We can, accordingly, interchange the blue and yellow wafers in G without changing any others; this makes b yellow, and we can put a blue wafer on W. Similarly, if we cannot pass from a to c, a belongs to an isolated group of red and green districts. Interchanging the wafers in these, a becomes green, and a red wafer can be put on W. Precisely similar reasoning applies in the case of five surrounding or surrounded districts, viz., e red, f blue, g green, h blue (two must, of course, be the same colour), and k yellow. Here either e belongs to an isolated group of red and green districts,

or k to one of yellow and green districts, or f to one of blue and yellow, and h to one of blue and red districts. In the first case, interchanging wafers as before, e becomes green, and a red wafer can be put on W; in the second, k becomes green, and a yellow wafer can be put on W; in the third, f becomes yellow and h red, and a blue wafer can be put on W. In all cases, before putting the wafer on W, we can interchange the colours of districts, *e.g.*, we can put red wafers in the place of all green ones, and *vice versâ*. Thus we can make the three colours adjacent to W any three we please. If, therefore, the districts adjacent to W belong to different groups of districts surrounding and surrounded by W, and so detached from each other, we can rearrange the wafers in each group, so that only three colours in all shall be adjacent to W, which can therefore have a wafer of the fourth colour placed on it. Thus in any case the district W can be wafered with a wafer of one of the four colours. Thus if the map can be coloured as developed at any stage, it can be coloured at the next. Hence, since it can obviously be coloured at the first stage when there is only one district, it can be coloured at the last."

The following extract from a communication made to Mr. Tucker by Mr. J. W. L. Glaisher, may be fitly placed here, as it bears upon points discussed in the present volume :—

"At the Meeting of the Society on February 9, 1879, after Professor Cayley had communicated his theorem

$$k'^2 - k^2k'^2s_1s_2s_3s_4 + k^2c_1c_2c_3c_4 - d_1d_2d_3d_4 = 0 \quad \ldots\ldots\ldots\ldots(\text{i.}),$$

where s_1, c_1, d_1, s_2, ... are the sn's, cn's, dn's of any four quantities u, v, r, s, whose sum is zero (*antè*, p. 43), I mentioned the following group of formulæ, also connecting the elliptic functions of four quantities whose sum is zero :—

$$\frac{s_1d_2 - s_2d_1}{c_1 - c_2} + \frac{s_3d_4 - s_4d_3}{c_3 - c_4} = 0 \quad \ldots\ldots\ldots\ldots(1),$$

$$\frac{s_1d_2 + s_2d_1}{c_1 + c_2} + \frac{s_3d_4 + s_4d_3}{c_3 + c_4} = 0 \quad \ldots\ldots\ldots\ldots(2),$$

$$\frac{s_1c_2 - s_2c_1}{d_1 - d_2} + \frac{s_3c_4 - s_4c_3}{d_3 - d_4} = 0 \quad \ldots\ldots\ldots\ldots(3),$$

$$\frac{s_1c_2 + s_2c_1}{d_1 + d_2} + \frac{s_3c_4 + s_4c_3}{d_3 + d_4} = 0 \quad \ldots\ldots\ldots\ldots(4),$$

$$\frac{c_1d_2 - c_2d_1}{s_1 - s_2} + \frac{c_3d_4 - c_4d_3}{s_3 - s_4} = 0 \quad \ldots\ldots\ldots\ldots(5),$$

$$\frac{c_1d_2 + c_2d_1}{s_1 + s_2} + \frac{c_3d_4 + c_4d_3}{s_3 + s_4} = 0 \quad \ldots\ldots\ldots\ldots(6),$$

which I had deduced from the formulæ

$$\frac{1 + \mathrm{cn}\,(u+v)}{\mathrm{sn}\,(u+v)} = -\frac{s_1d_2 - s_2d_1}{c_1 - c_2} = \frac{c_1 + c_2}{s_1d_2 + s_2d_1} \quad \ldots\ldots\ldots(7),$$

$$\frac{1 - \mathrm{cn}\,(u+v)}{\mathrm{sn}\,(u+v)} = \frac{s_1d_2 + s_2d_1}{c_1 + c_2} = -\frac{c_1 - c_2}{s_1d_2 - s_2d_1} \quad \ldots\ldots\ldots(8),$$

$$\frac{1+\mathrm{dn}(u+v)}{\mathrm{sn}\,(u+v)} = -\,k^2\,\frac{s_1c_2-s_2c_1}{d_1-d_2} = \frac{d_1+d_2}{s_1c_2+s_2c_1} \quad \ldots\ldots\ldots(9),$$

$$\frac{1-\mathrm{dn}\,(u+v)}{\mathrm{sn}\,(u+v)} = k^2\,\frac{s_1c_2+s_2c_1}{d_1+d_2} = -\,\frac{d_1-d_2}{s_1c_2-s_2c_1} \quad \ldots\ldots\ldots(10),$$

$$\frac{\mathrm{dn}\,(u+v)-\mathrm{cn}\,(u+v)}{\mathrm{sn}\,(u+v)} = -\,\frac{c_1d_2-c_2d_1}{s_1-s_2} = k'^2\,\frac{s_1+s_2}{c_1d_2+c_2d_1} \ldots\ldots\ldots(11),$$

$$\frac{\mathrm{dn}\,(u+v)+\mathrm{cn}\,(u+v)}{\mathrm{sn}\,(u+v)} = \frac{c_1d_2+c_2d_1}{s_1+s_2} = -\,k'^2\,\frac{s_1-s_2}{c_1d_2-c_2d_1} \ldots\ldots\ldots(12).$$

The formulæ (1) ... (6) follow at once from these, since, when $u+v+r+s=0$, we have

$$\mathrm{sn}\,(u+v) = -\,\mathrm{sn}\,(r+s),\ \mathrm{cn}\,(u+v) = \mathrm{cn}\,(r+s),\ \mathrm{dn}\,(u+v) = \mathrm{dn}\,(r+s).$$

When, at the following Meeting on February 13, Professor H. J. S. Smith gave his five formulæ, (i.) ... (v.), (p. 97), connecting the elliptic functions of four quantities whose sum is zero, I remarked that those which I had mentioned at the former Meeting were probably deducible from (v.); and this is the case, for, multiplying up in (1) and (2), these equations become

$$s_1c_3d_2-s_2c_3d_1-s_1c_4d_2+s_2c_4d_1+s_3c_1d_4-s_4c_1d_3-s_3c_2d_4+s_4c_2d_3=0,$$

$$s_1c_3d_2+s_2c_3d_1+s_1c_4d_2+s_2c_4d_1+s_3c_1d_4+s_4c_1d_3+s_3c_2d_4+s_4c_2d_3=0;$$

which give, by addition and subtraction,

$$s_1c_3d_2+s_2c_4d_1+s_3c_1d_4+s_4c_2d_3=0,$$

$$s_2c_3d_1+s_1c_4d_2+s_4c_1d_3+s_3c_2d_4=0;$$

so that (1) ... (6) are equivalent to the group of six formulæ, of which Professor Smith's formula (v.) is the type.

If, instead of combining the corresponding expressions involving the suffixes 1, 2 and 3, 4 in (7) ... (12), the dissimilar expressions be combined, we are led to Professor Smith's formulæ (ii.), (iii.), (iv.). For, from either (7) or (8),

$$\frac{s_1d_2-s_2d_1}{c_1-c_2}-\frac{c_3+c_4}{s_3d_4+s_4d_3}=0,$$

$$\frac{s_3d_4-s_4d_3}{c_3-c_4}-\frac{c_1+c_2}{s_1d_2+s_2d_1}=0,$$

which, multiplying up, give, by addition and subtraction,

$$s_1s_3d_2d_4-s_2s_4d_1d_3-c_1c_3+c_2c_4=0,$$

$$s_1s_4d_2d_3-s_2s_3d_1d_4-c_1c_4+c_2c_3=0;$$

that is, they give (iv.); and similarly from (9) or (10), we obtain (ii.), and from (11) or (12), (iii.).

The formulæ (7) ... (12) thus afford an elementary demonstration of Professor Smith's results (ii.) ... (v.). They do not, however, lead directly to Professor Cayley's original equation (i.); but this can be deduced algebraically from any two of the equations (ii.).

For, taking (ii.) and (iii.), viz.,

$$d_1 d_2 - d_3 d_4 = k^2 (s_1 s_2 c_3 c_4 - s_3 s_4 c_1 c_2),$$
$$k'^2 (s_1 s_2 - s_3 s_4) = c_1 c_2 d_3 d_4 - c_3 c_4 d_1 d_2,$$

and multiplying together the right-hand members and the left-hand members of these equations, we obtain, after reduction,

$$(s_1 s_2 d_3 d_4 + s_3 s_4 d_1 d_2)(k^2 k'^2 s_1 s_2 s_3 s_4 + d_1 d_2 d_3 d_4)$$
$$= (s_1 s_2 d_3 d_4 + s_3 s_4 d_1 d_2)(k'^2 + k^2 c_1 c_2 c_3 c_4),$$

and therefore $k^2 k'^2 s_1 s_2 s_3 s_4 + d_1 d_2 d_3 d_4 = k'^2 + k^2 c_1 c_2 c_3 c_4$.

If we had multiplied the members of the two equations cross-wise, the factor $c_1 c_2 + c_3 c_4$ would have divided out, leaving the formula (i.). Similar results occur on multiplying the members of the equations (ii.) and (iv.), or (iii.) and (iv.), the factors

$$c_1 c_2 d_3 d_4 + c_3 c_4 d_1 d_2, \quad s_1 s_2 c_3 c_4 + s_3 s_4 c_1 c_2, \quad s_1 s_2 + s_3 s_4, \quad d_1 d_2 + d_3 d_4$$

dividing out in the respective cases, and leaving (i.) in each case.

I have also obtained proofs by ordinary elliptic-function theorems of Professor Smith's eleven formulæ (pp. 95, 96) relating to the general case in which the arguments are unrestricted.

Professor Cayley's theorem (i.) was, I find, given by Gudermann in Crelle's Journal, t. xviii., p. 164."

It has been pointed out to the Secretaries that the first of Dr. Zeuthen's Geometrical Theorems (pp. 196—203) may be obtained immediately by reciprocating one of the two conics into the other; the four different ways in which this can be done corresponding to the four different ways in which the four common tangents can be correlated to the four points of intersection. This first geometrical theorem serves, ofcourse, no less than the second, to demonstrate Dr. Zeuthen's Algebraical principle.

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